Formula 1: Sum of sets

When $\mathrm{A} = \{a_{1,} a_2 \cdots a_l, c_1, c_2, \cdots c_n\}$ and $\mathrm{B} = \{b_1, b_2, \cdots b_m, c_1, c_2, \cdots c_n, \}$,

$$D = A \cup B = \{a_{1,} a_2 \cdots a_l, b_1, b_2, \cdots b_m, c_1, c_2, \cdots c_m\}$$

Formula 2: Product set

When $\mathrm{A} = \{a_{1,} a_2 \cdots a_l, c_1, c_2, \cdots c_n\}$ and $\mathrm{B} = \{b_1, b_2, \cdots b_m, c_1, c_2, \cdots c_n, \}$

$$\mathrm{C} = \mathrm{A} \cap \mathrm{B} = \{ c_1, c_2, \cdots c_m\}$$

Formula 3: Probability of sum of sets

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$$

Formula 4: Probability of product set

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$$

Formula 5: Probability of sequential phenomena

In the case A gives no effect to $\mathrm{P}(B)$

$$\mathrm{P}(B|A) = \mathrm{P}(A)\mathrm{P}(|A) = \mathrm{P}(A)\mathrm{P}(B)$$

Formula 6: Binominal coefficient

$$ {}_nC_r = \frac{n \cdot (n-1) \cdot \cdots \cdot \big(n - (n-k)\big)}{k \cdot (k-1) \cdot \cdots \cdot 1} = \frac{n!}{k!\,(n-k)!}$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Formula 7: Binomial probability

$$p(k) = \quad {}_nC_k p^k (1-p)^{n-k}$$

$$p(k) = \binom{n}{k} \mathrm{p}^{n-k} (1-p)^k$$

Formula 8: Expansion of binomial

$$(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} \mathrm{p}^{n-k} q^k$$

Formula 9: Expectation value

$$E\big(f(x)\big) = \textstyle\sum_{i=1}^{n} f(x_i)\, p_i$$

Where

$f(x_i)$: obtainable value by event i

$p(i)$: possibility of event i

$E\big(f(x)\big)$: expectation value

Formula 10: average

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \sum_{i=1}^{n} x_i p_i$$

$\bar{x}$: average of x

p_i: probability of i

Formula 11: Sum of square of the distance of data from average (SS).

$$SS = \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$\bar{x}$: average of data

n: number of data

Formula 12: Logarithm of binomial coefficient

$$\log W(x) = \log(n!) - \log(x!) - \log(n-x)! + k\log(p) + (n-x)\log(q)$$

Formula 13: Simplified calculation of variance

$$V_x = E(x^2) - E(x)^2$$

V_x: variance of x

Formula 14: Quadratic moment of sample average around average of parent population

$$E((M-\mu)^2) = \frac{\sigma^2}{n}$$

Formula 15: Standard error

$$S.E = \sqrt{E(M^2)} = \frac{\sigma}{\sqrt{n}}$$

Formula 16: Poisson distribution

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Formula 17: Normal distribution

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Formula 18: Γ function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

z: complex number, $z \geq 0$ on complex plane

Formula 19: Nature of Γ function

$$\Gamma(n+1) = n!$$

$$\Gamma(1) = 1$$

$$\Gamma\left(\frac{1}{2}\right) = \pi$$

$$\Gamma\left(\frac{1}{2} + n\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$$

Formula 20: χ^2 distribution

$$\mathrm{P}({\chi^2}_{\phi}) = \frac{{{\chi^2}_{\phi}}^{\frac{\phi}{2}-1}}{2^{\frac{\phi}{2}}\Gamma\left(\frac{\phi}{2}\right)} e^{\frac{-{\chi^2}_{\phi}}{2}}$$

ϕ: degree of freedom

Formula 21: χ^2 value

$$\chi^2 = \sum \frac{(f_i - e_i)^2}{e_i}$$

f: observed value, e:expectation value

Formula 22: β function

$$\beta(p,q)) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Formula 23: Student's t distribution

$$S(t) = \frac{1}{\sqrt{n}\beta\left(\frac{n}{2},\frac{1}{2}\right)\left(\frac{t^2}{n^2}+1\right)^{\frac{n}{2}+1}}$$

Formula 24: Student's t value

$$\mathrm{t} = \frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}}$$

Formula 25: F distribution

$$P(F) = \frac{{n_1}^{\frac{n_1}{2}}{n_2}^{\frac{n_2}{2}}}{\beta(n_1,n_2)} \cdot \frac{F^{\frac{n_1}{2}-1}}{(n_1F+n_2)^{\frac{n_1+n_2}{2}}}$$

Formula 26: Definition of derivation

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$f'(x)$: derivate of $f(x)$

Formula 27: Law of mean

$$\frac{f(b)-f(a)}{b-a} = f'(c), \quad (a \le c \le b)$$

Formula 28: Expansion of law of mean

$$f(x) = f(a) + (x-a)f'(c_1)$$

Formula 29: Secondary expansion of law of mean

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a)$$

Formula 30: Taylor expansion (1)

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a)$$

Formula31: Taylor expansion (2)

$$f(x) \fallingdotseq f(a) + \frac{(x-a)}{1} f'(a) + \frac{(x-a)^2}{2*1} f''(a) + \frac{(x-a)^3}{3*2*1} f'''(a) \quad + \qquad + \frac{(x-a)^n}{n!} f^{(n)}(a)$$

Formula 32: differentiation of multiplicated function

$$\{g(x)f(x)\}' = g(x)f'(x) + g'(x)f(x)$$

Formula 33: Partial integration

$$\int g'(x)f(x)dx = g(x)f(x) - \int g(x)f'(x)dx$$

Formula 34: Napier's constant (1)

$$e = \lim_{\mathrm{n}\to\infty} \left(1 + \frac{1}{\mathrm{n}}\right)^{\mathrm{n}}$$

Formula 35: Differentiation of logarithm

$$\frac{d \log_e x}{dx} = \frac{1}{x}$$

Formula 36: Differentiation of exponential

$$\frac{df(x)}{dx} = f(x)$$

Formula 37: Napier's constant (2)

$$1 + \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k!} = e$$

Formula 38: Structure of data

$$x_i = M + e_i$$

$$M: mean$$

$$e_i: deviation\ from\ mean$$

Formula 39: total SS

$$SS_{total} = nSS_A + mSS_B$$

Formula 40: Variance sum of two different sample populations (combined variance)

$${\sigma_{A+B}}^2 = \frac{(m-1){\sigma_A}^2 + (n-1){\sigma_B}^2}{m+n-2}$$

Formula 41: Variance of difference (combined variance)

$${\sigma_{A-B}}^2 = \frac{(m-1){\sigma_{\hat{A}}}^2 + (n-1){\sigma_{\hat{B}}}^2}{m+n-2}$$

Formula 42: Paired t value

$$\mathrm{t} = \frac{M_c}{\frac{\sigma_c}{\sqrt{n}}}$$

Formula 43: Unpaired t value

$$\mathrm{t} = \frac{M_A - M_B}{\sigma_{A-B}\sqrt{\frac{1}{n_A} + \frac{1}{n_B}}}$$

Formula 44: Contribution ration

$$r^2 = \frac{{SS_{xy}}^2}{SS_x SS_y}$$

r^2: contribution ratio

Formula 45: Coefficient of correlation

$$r = \frac{SS_{xy}}{\sqrt{SS_x}\sqrt{SS_y}}$$

Formula 46: Cauchy-Schwarz inequation

$$\sqrt{\alpha^2 + \beta^2 + \gamma^2}\sqrt{\delta^2 + \varepsilon^2 + \zeta^2} \geq \alpha\delta + \beta\varepsilon + \gamma\,\zeta$$

Formula 47: Observed χ^2 value

$$\chi^2 = \sum \frac{(f_i - e_i)^2}{e_i}$$

f_i: observed data in sub sample population i

e_i: expectation value of f_i

Formula 48: Sum of matrixes

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2j} & & a_{2n} \\ \vdots & & & \vdots & & \vdots \\ a_{i1} & & & a_{ij} & \cdots & a_{in} \\ \vdots & & & \vdots & & \vdots \\ a_{n1} & a_{2n} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & & b_{2j} & & b_{2n} \\ \vdots & & & \vdots & & \vdots \\ b_{i1} & & & b_{ij} & \cdots & b_{in} \\ \vdots & & & \vdots & & \vdots \\ b_{n1} & b_{2n} & \cdots & b_{nj} & \cdots & b_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & a_{ij}+b_{ij} & \vdots \\ a_{n1}+b_{n1} & \cdots & +a_{nn} + b_{nn} \end{pmatrix}$$

Formula 49: multiplication of scalar to matrix

$$\alpha \begin{pmatrix} a_{11} & \cdots & a_n \\ \vdots & a_{ij} & \vdots \\ a_{n1} & \cdots & +a_{nn} \end{pmatrix} = \begin{pmatrix} \alpha a_{11} & \cdots & \alpha a_n \\ \vdots & \alpha a_{ij} & \vdots \\ \alpha a_{n1} & \cdots & \alpha a_{nn} \end{pmatrix}$$

Formula 50: Product of matrixes

$$
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\
a_{21} & a_{22} & & a_{2j} & & a_{2n} \\
\vdots & & & & & \vdots \\
a_{i1} & & & a_{ij} & & a_{in} \\
\vdots & & & & & \vdots \\
a_{n1} & a_{2n} & \cdots & a_{nj} & \cdots & a_{nn}
\end{pmatrix}
\begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\
b_{21} & b_{22} & & b_{2j} & & b_{2n} \\
\vdots & & & \vdots & & \vdots \\
b_{i1} & & & b_{ij} & \cdots & b_{in} \\
\vdots & & & \vdots & & \vdots \\
b_{n1} & b_{2n} & \cdots & b_{nj} & \cdots & b_{nn}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
a_{11}b_{11} + \cdots a_{1k}b_{k1} \cdots + a_{1n}b_{n1} & \cdots & a_{11}b_{1n} + \cdots a_{1k}b_{kn} \ldots a_{1n}b_{nn} \\
\vdots & a_{i1}b_{1j} + \cdots a_{ik}b_{kj} \cdots + a_{in}b_{nj} & \vdots \\
a_{n1}b_{11} + \cdots a_{nk}b_{k1} \cdots + a_{nn}b_{n1} & \cdots & a_{n1}b_{1n} + \cdots a_{nk}b_{kn} \cdots + a_{nn}b_{nn}
\end{pmatrix}
$$

Formula 51: Order of matrix calculation

$$\boldsymbol{AB} \neq \boldsymbol{BA}$$

$$(\mathbf{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$$

Formula 52: Cofactor matrix (3×3)

When

$$\boldsymbol{A} = \begin{pmatrix} a & b & c \\ k & l & m \\ s & t & u \end{pmatrix}$$

$$\widetilde{\boldsymbol{A}} = \begin{pmatrix}
\begin{vmatrix} l & m \\ t & u \end{vmatrix} & -\begin{vmatrix} b & c \\ t & u \end{vmatrix} & \begin{vmatrix} b & c \\ l & m \end{vmatrix} \\
-\begin{vmatrix} k & m \\ s & u \end{vmatrix} & \begin{vmatrix} a & c \\ s & u \end{vmatrix} & -\begin{vmatrix} a & c \\ k & m \end{vmatrix} \\
\begin{vmatrix} k & l \\ s & t \end{vmatrix} & -\begin{vmatrix} a & b \\ s & t \end{vmatrix} & \begin{vmatrix} a & b \\ k & l \end{vmatrix}
\end{pmatrix}$$

Formula 53: Inverse matrix

$$\boldsymbol{A}^{-1} = \frac{\widetilde{\boldsymbol{A}}}{|\boldsymbol{A}|}$$

Formula 54: inverse matrix (3×3)

When

$$\boldsymbol{A} = \begin{pmatrix} a & b & c \\ k & l & m \\ s & t & u \end{pmatrix}$$

,

$$\boldsymbol{A}^{-1} = \frac{1}{\begin{vmatrix} a & b & c \\ k & l & m \\ s & t & u \end{vmatrix}} \begin{pmatrix}
\begin{vmatrix} l & m \\ t & u \end{vmatrix} & -\begin{vmatrix} b & c \\ t & u \end{vmatrix} & \begin{vmatrix} b & c \\ l & m \end{vmatrix} \\
-\begin{vmatrix} k & m \\ s & u \end{vmatrix} & \begin{vmatrix} a & c \\ s & u \end{vmatrix} & -\begin{vmatrix} a & c \\ k & m \end{vmatrix} \\
\begin{vmatrix} k & l \\ s & t \end{vmatrix} & -\begin{vmatrix} a & b \\ s & t \end{vmatrix} & \begin{vmatrix} a & b \\ k & l \end{vmatrix}
\end{pmatrix}$$

Formula 55: Cofactor expansion

$$\sum_{i=1}^{n} a_{ij} a^{ij} = a_{1j} a^{1j} + a_{2j} a^{2j} + \cdots + a_{nj} a^{nj} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$j = 1, 2, \cdots, n$$

Formula 56: Cramer's rule

When

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad \boldsymbol{X}=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}. \quad \boldsymbol{b}=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

and

$$\boldsymbol{AX}=\boldsymbol{b}$$

,

$$x_i=\frac{|\boldsymbol{A_i}|}{|\boldsymbol{A}|}$$

Here, definition of $\boldsymbol{A_i}$ is as follow.

$$\boldsymbol{A_i}=\begin{pmatrix} a_{1,1} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \cdots & a_{1,n} \\ a_{2,1} & \cdots & a_{2,i-1} & b_2 & a_{2,i+1} & \cdots & a_{2,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,i-1} & b_n & a_{n,i+1} & \cdots & a_{n,n} \end{pmatrix}$$

Formula 57: Eigen Formula

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}=0$$

Formula 58: Products of partial matrixes

$$\boldsymbol{A}=\begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{pmatrix}$$

$$\boldsymbol{A}=\left(\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1q} & a_{1q+1} & \cdots & a_{1n} \\ \vdots & A_{11} & \vdots & \vdots & A_{12} & \vdots \\ a_{p1} & \cdots & a_{pq} & a_{pq+1} & \cdots & a_{pn} \\ \hline a_{p+11} & \cdots & a_{p+11} & a_{p+1\,q+1} & \cdots & a_{p+1\,q+1} \\ \vdots & A_{21} & \vdots & \vdots & A_{22} & \vdots \\ a_{n1} & \cdots & a_{nq} & a_{nq+1} & \cdots & a_{nn} \end{array}\right)$$

$$\boldsymbol{B}=\begin{pmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{pmatrix}$$

$$\boldsymbol{B}=\left(\begin{array}{ccc|ccc} b_{11} & \cdots & b_{1p} & b_{1p+1} & \cdots & b_{1n} \\ \vdots & B_{11} & \vdots & \vdots & B_{12} & \vdots \\ b_{q1} & \cdots & b_{qp} & b_{qp+1} & \cdots & b_{qn} \\ \hline b_{q+1\,1} & \cdots & b_{q+1\,p} & b_{q+1p+1} & \cdots & b_{q+1n} \\ \vdots & B_{21} & \vdots & \vdots & B_{22} & \vdots \\ b_{n1} & \cdots & b_{n\,p} & b_{n\,p+1} & \cdots & b_{n\,n} \end{array}\right)$$

$$\boldsymbol{AB}=\begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{pmatrix}\begin{pmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{pmatrix}=\begin{pmatrix} \boldsymbol{A}_{11}\boldsymbol{B}_{11}+\boldsymbol{A}_{12}\boldsymbol{B}_{21} & \boldsymbol{A}_{11}\boldsymbol{B}_{12}+\boldsymbol{A}_{12}\boldsymbol{B}_{22} \\ \boldsymbol{A}_{21}\boldsymbol{B}_{11}+\boldsymbol{A}_{22}\boldsymbol{B}_{21} & \boldsymbol{A}_{21}\boldsymbol{B}_{12}+\boldsymbol{A}_{22}\boldsymbol{B}_{22} \end{pmatrix}$$

$$\boldsymbol{A_{11}B_{11}} + \boldsymbol{A_{12}B_{21}}: p \times p \text{ square matrix}$$

$$\boldsymbol{A_{21}B_{12}} + \boldsymbol{A_{22}B_{22}}: (n-p) \times (n-p) \text{ square matrix}$$

Formula 59: Similarity of matrixes

$$\boldsymbol{C} = \boldsymbol{P^{-1}DP}$$

Formula 60:　Diagonalization

$$\boldsymbol{Q^{-1}CQ} = \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_p \end{pmatrix}$$

Formula 61: Spectral decomposition

$$\boldsymbol{A} = \lambda_1 \boldsymbol{e_1 e_1}^T + \lambda_2 \boldsymbol{e_2 e_2}^T + \cdots + \lambda_p \boldsymbol{e_p e_p}^T$$

or

$$\boldsymbol{A} = \sum_{i=1}^{p} \lambda_i \boldsymbol{e_i e_i}^T$$

$\boldsymbol{e_i}$: unit vector of eigenvector,

$$\boldsymbol{e_i} \perp \boldsymbol{e_j}$$

Formula 62: Power method of matrix

$$\boldsymbol{A^m} = \boldsymbol{P\Lambda^m P^{-1}}$$

Formula 63: Maximum and minimum in quadratic form

When symmetric matrix $\boldsymbol{A}$ is positive definite

$$\max_{\boldsymbol{x \neq 0}} \frac{\boldsymbol{x}^T \boldsymbol{Ax}}{\boldsymbol{x}^T \boldsymbol{x}} = \lambda_1$$

Similarly,

$$\min_{\boldsymbol{x \neq 0}} \frac{\boldsymbol{x}^T \boldsymbol{Ax}}{\boldsymbol{x}^T \boldsymbol{x}} = \lambda_p$$

$$\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_p \end{pmatrix}$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0$$

Formula 64: Cauchy Schwarz's inequation

$$\left(a_1 b_1 + a_2 b_2 + \cdots + a_p b_p\right)^2 \leq \left({a_1}^2 + {a_2}^2 + \cdots + {a_p}^2\right)\left({b_1}^2 + {b_2}^2 + \cdots + {b_p}^2\right)$$

Formula 65: Expansion of Cauchy Schwarz's inequation

$$(\boldsymbol{\alpha}^T \boldsymbol{B\alpha})(\boldsymbol{\beta}^T \boldsymbol{B^{-1}\beta}) \geq (\boldsymbol{\alpha^T \beta})^2$$

When

$$\boldsymbol{B}^{\frac{1}{2}} \boldsymbol{\alpha} = c \boldsymbol{B}^{-\frac{1}{2}} \boldsymbol{\beta E} \ ,$$

$$(\boldsymbol{\alpha}^T\boldsymbol{B}\boldsymbol{\alpha})(\boldsymbol{\beta}^T\boldsymbol{B}^{-1}\boldsymbol{\beta}) \geq (\boldsymbol{\alpha}^T\boldsymbol{\beta})^2$$

Formula 66 Method of Lagrange multiplier

$$\frac{\partial L(\boldsymbol{x},\lambda)}{\partial(\boldsymbol{x},\lambda)} = \begin{pmatrix} \frac{\partial L(\boldsymbol{x},\lambda)}{\partial x_1} \\ \vdots \\ \frac{\partial L(\boldsymbol{x},\lambda)}{\partial x_p} \\ \frac{\partial L(\boldsymbol{x},\lambda)}{\partial \lambda} \end{pmatrix}$$

$$L(\boldsymbol{x},\lambda) = f(\boldsymbol{x}) - \lambda g(\boldsymbol{x})$$

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{x_1} \\ \vdots \\ \boldsymbol{x_p} \end{pmatrix}$$

Formula 67: Variance and covariance matrix

$$\boldsymbol{\Sigma} = \frac{1}{n}\begin{pmatrix} \sum_{k=1}^{n} c_{1k}c_{1k} & \sum_{k=1}^{n} c_{1k}c_{2k} & \cdots & \sum_{k=1}^{n} c_{1k}c_{pk} \\ \sum_{k=1}^{n} c_{2k}c_{1k} & \sum_{k=1}^{n} c_{2k}c_{2k} & \cdots & \sum_{k=1}^{n} c_{2k}c_{pk} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} c_{pk}c_{1k} & \sum_{k=1}^{n} c_{pk}c_{2k} & \cdots & \sum_{k=1}^{n} c_{pk}c_{pk} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{p2} \end{pmatrix}$$

Formula 68: Correlation matrix

$$\boldsymbol{\rho} = \begin{pmatrix} \frac{\sigma_{11}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{11}}} & \frac{\sigma_{12}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}} & \cdots & \frac{\sigma_{1p}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{pp}}} \\ \frac{\sigma_{21}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{11}}} & \frac{\sigma_{22}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{22}}} & \cdots & \frac{\sigma_{2p}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{pp}}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sigma_{p1}}{\sqrt{\sigma_{pp}}\sqrt{\sigma_{11}}} & \frac{\sigma_{p2}}{\sqrt{\sigma_{pp}}\sqrt{\sigma_{22}}} & \cdots & \frac{\sigma_{pp}}{\sqrt{\sigma_{pp}}\sqrt{\sigma_{pp}}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1p} \\ \rho_{21} & 1 & \cdots & \rho_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{p1} & \rho_{p2} & \cdots & 1 \end{pmatrix}$$

Formula 69: Relation among variance covariance matrix ($\boldsymbol{\Sigma}$), correlation matrix ($\boldsymbol{\rho}$) and variance matrix ($\boldsymbol{V}$).

$$\boldsymbol{V}^{\frac{1}{2}}\boldsymbol{\rho}\boldsymbol{V}^{\frac{1}{2}} = \boldsymbol{\Sigma}$$

$$\boldsymbol{V}^{-\frac{1}{2}}\boldsymbol{\Sigma}\boldsymbol{V}^{-\frac{1}{2}} = \boldsymbol{\rho}$$

Formula 70: Mahalanobis' Distance

$$D_{\boldsymbol{a}-\boldsymbol{b}} = \sqrt{(\boldsymbol{a}-\boldsymbol{b})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{a}-\boldsymbol{b})}$$

$$(\boldsymbol{\Sigma}: variance\ covariance\ matrix)$$

Formula 71: Pseudo-inverse matrix

$$\boldsymbol{X}^{\#} = (\boldsymbol{X}^T \boldsymbol{X})^{-1} \boldsymbol{X}^T$$

Formula 72: Singular value decomposition

$$\boldsymbol{M} = \boldsymbol{U\Sigma V}^T$$

$$\text{when } p < n$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} \gamma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \gamma_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \gamma_p & 0 & \cdots & 0 \end{pmatrix}_{p \times n}$$

$$\text{when } p > n$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} \gamma_1 & 0 & \cdots & 0 \\ 0 & \gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \gamma_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{p \times n}$$

Formula 73: multiple linear regression

$$\boldsymbol{Y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad \boldsymbol{X}_{+1} = \begin{pmatrix} 1 & x_{11} & \cdots & x_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & \cdots & x_{np} \end{pmatrix}, \quad \boldsymbol{A}_{+1} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_p \end{pmatrix}, \quad \boldsymbol{E} = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

$$\boldsymbol{Y} = \boldsymbol{X}_{+1}\boldsymbol{A}_{+1} + \boldsymbol{E}$$

$$\boldsymbol{A}_{+1} = \boldsymbol{X}_{+1}{}^{\#}\boldsymbol{Y}$$

Formula 74: Partial correlation

$$\boldsymbol{R}^{-1} = \frac{1}{|\boldsymbol{R}|} \begin{pmatrix} \frac{SS_{yy}SS_{zz} - SS_{yz}{}^2}{SS_{yy}SS_{zz}} & \frac{SS_{yz}SS_{zx} - SS_{xy}SS_{zz}}{\sqrt{SS_{yy}}\sqrt{SS_{xx}}SS_{zz}} & \frac{SS_{xy}SS_{yz} - SS_{xz}SS_{yy}}{\sqrt{SS_{xx}}\sqrt{SS_{zz}}SS_{yy}} \\ \frac{SS_{yz}SS_{zx} - SS_{xy}SS_{zz}}{\sqrt{SS_{yy}}\sqrt{SS_{xx}}SS_{zz}} & \frac{SS_{zz}SS_{xx} - SS_{zx}{}^2}{SS_{zz}SS_{xx}} & \frac{SS_{xy}SS_{zx} - SS_{yz}SS_{xx}}{\sqrt{SS_{yy}}\sqrt{SS_{zz}}SS_{xx}} \\ \frac{SS_{xy}SS_{yz} - SS_{xz}SS_{yy}}{\sqrt{SS_{xx}}\sqrt{SS_{zz}}SS_{yy}} & \frac{SS_{xy}SS_{zx} - SS_{yz}SS_{xx}}{\sqrt{SS_{yy}}\sqrt{SS_{zz}}SS_{xx}} & \frac{SS_{xx}SS_{yy} - SS_{xy}{}^2}{SS_{xx}SS_{yy}} \end{pmatrix}$$

$$= \begin{pmatrix} r^{xx} & r^{xy} & r^{xz} \\ r^{yx} & r^{yy} & r^{yz} \\ r^{zx} & r^{yz} & r^{zz} \end{pmatrix}$$

$$\mathrm{r}_{xy/z} = \frac{-r^{xy}}{\sqrt{r^{xx}}\sqrt{r^{yy}}}$$

$$\mathrm{r}_{ij/rest} == \frac{-r^{ij}}{\sqrt{r^{ii}}\sqrt{r^{jj}}}$$

Formula 75: Linear discrimination analysis

$$\boldsymbol{M} = \begin{pmatrix} \sum_{k=1}^{m} n_k\, {\mu_{k1}}^2 & \sum_{k=1}^{m} n_k \mu_{k1}\mu_{k2} & \cdots & \sum_{k=1}^{m} n_k \mu_{k1}\mu_{k2} \\ \sum_{k=1}^{m} n_k\, \mu_{k2}\mu_{k1} & \sum_{k=1}^{m} n_k {\mu_{k2}}^2 & \cdots & \sum_{k=1}^{m} n_k \mu_{k2}\mu_{kp} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{m} n_k\, \mu_{kp}\mu_{k1} & \sum_{k=1}^{m} n_k \mu_{kp}\mu_{k2} & \cdots & \sum_{k=1}^{m} n_k\, {\mu_{kp}}^2 \end{pmatrix}$$

$$\boldsymbol{V} = \begin{pmatrix} \sum_{\boldsymbol{k=1}}^{\boldsymbol{m}} \sum_{i=1}^{n_k} {m_{ki}}^2 & \sum_{\boldsymbol{k=1}}^{\boldsymbol{m}} \sum_{i=1}^{n_k} m_{ki1} m_{ki2} & \cdots & \sum_{\boldsymbol{k=1}}^{\boldsymbol{m}} \sum_{i=1}^{n_k} m_{ki}\; m_{kip} \\ \sum_{\boldsymbol{k=1}}^{\boldsymbol{m}} \sum_{i=1}^{n_k} m_{ki2} m_{ki1} & \sum_{\boldsymbol{k=1}}^{\boldsymbol{m}} \sum_{i=1}^{n_k} {m_{ki2}}^2 & \cdots & \sum_{\boldsymbol{k=1}}^{\boldsymbol{m}} \sum_{i=1}^{n_k} m_{ki2} m_{kip} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{\boldsymbol{k=1}}^{\boldsymbol{m}} \sum_{i=1}^{n_k} m_{kip} m_{ki} & \sum_{\boldsymbol{k=1}}^{\boldsymbol{m}} \sum_{i=1}^{n_k} m_{kip} m_{ki2} & \cdots & \sum_{\boldsymbol{k=1}}^{\boldsymbol{m}} \sum_{i=1}^{n_k} {m_{kip}}^2 \end{pmatrix}$$

$$f(\boldsymbol{A}) = \left(\frac{SS_{subpopulatoi}}{\sum_{k=1}^{m} SS_k}\right) = \frac{\boldsymbol{A}^T \boldsymbol{M}\boldsymbol{A}}{\boldsymbol{A}^T \boldsymbol{V}\boldsymbol{A}}$$

$$\frac{df(\boldsymbol{A})}{d\boldsymbol{A}} = 0$$

Formula 76: Centralizing matrix

$$\boldsymbol{G} = \begin{pmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & 1-\frac{1}{n} & \cdots & -\frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & 1-\frac{1}{n} \end{pmatrix}$$

Formula 77: Multidimensional scaling method

$$-\frac{1}{2}\boldsymbol{G}_n \boldsymbol{D}^2 \boldsymbol{G}_n = \boldsymbol{P}\boldsymbol{\Lambda}^{\frac{1}{2}}\left(\boldsymbol{P}\boldsymbol{\Lambda}^{\frac{1}{2}}\right)^{\mathrm{T}}$$

$$\boldsymbol{Y} = \boldsymbol{P}\boldsymbol{\Lambda}^{\frac{1}{2}}$$

Formula 78: Jensen's inequality,

$$f\big(Ex(\boldsymbol{x})\big) \geq Ex\big(f(x)\big)$$

Formula 79：Orthomax standard

$$\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1p} \\ \vdots & \ddots & \vdots \\ \lambda_{m1} & \cdots & \lambda_{mp} \end{pmatrix}_{m\times p}$$

$$\mathrm{Q}_{or} = \sum_{j=1}^{p}\sum_{k=1}^{m}{\lambda_{jk}}^4 - \frac{\omega}{m}\sum_{k=1}^{p}\left(\sum_{j=1}^{m}{\lambda_{jk}}^2\right)^2$$

ω: weight

Formula 80：Oblimin standard

$$\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1p} \\ \vdots & \ddots & \vdots \\ \lambda_{m1} & \cdots & \lambda_{mp} \end{pmatrix}_{m\times p}$$

$$\mathrm{Q}_{ob} = \sum_{k<l=1}^{p}\left\{\sum_{j=1}^{m}{\lambda_{jk}}^2{\lambda_{jl}}^2 - \frac{\omega}{m}\left(\sum_{j=1}^{m}{\lambda_{jk}}^2\right)\left(\sum_{j=1}^{m}{\lambda_{jl}}^2\right)\right\}$$

ω: weight