III-2. Probability distribution

III-2-1. Frequentism and binomial distribution

When we accept frequentism, we can draw the shape of distribution of probability of phenomena which appear accurately following the philosophy.

For the understanding of calculation of binomial distribution, the author should explain concept of set and meaning of combination.

"Set" is a cohesion of elements. In the case when elements of set A are 1,3,4,6,7, the set is expressed as $A = \{1.3.4.6,7\}$. In this notation, {dog} means all dogs including your dog, the dog next door, St. Bernard dog, Chihuahua and so on, and {animal} means all animal including {dog} and {cat}. When all the elements of set B is contained in set A, we say that B is included in A, or that B is subset of A. This relation is expressed as $B \sqsubseteq A$. When $B \sqsubseteq A$ and $B \neq A$, we say that B is proper subset of A, and the relation is expressed as $B \subset A$. As an example, {dog} and {cat} are subsets of {animal}, and the relation is expressed as {dog} ⊂ {animal} and {cat} ⊂ {animal}. Such relations are called inclusion. In case of $A \subset B \subset C$, $A \subset C$ is intuitively obvious. When an element e is included in a set A, the relation is expressed as e ∈ A, meaning that e is a element of set A or that the set A includes e as an element. Using the sign ∈, we can express that the dog in the next door ∈ dog. When all the elements of set $C=\{c_1, c_2, \cdots c_n\}$ are included in a set A and B. This is expressed as $e_1, e_2, \cdots e_n \in A$ and $e_1, e_2, \cdots e_n \in B$ using ∈, and set C is called a product set of A and B, expressed as $C = A \cap B$. We read A ∩ B as A and B. In the case when $A = \{dog\}$ and $B = \{female\}$, then $A \cap B = \{female\ dog\}$. A set D made by union of a number of different sets (set A and set B) is sum set of A and B. The relation is expressed as $D = A \cup B$.

If

$$A = \{a_{1,} a_2 \cdots a_l, c_1, c_2, \cdots c_n\} \text{ and } B = \{b_1, b_2, \cdots b_m, c_1, c_2, \cdots c_n, \ \},$$

Then

$$D = A \cup B = \{a_{1,} a_2 \cdots a_l, b_1, b_2, \cdots b_m, c_1, c_2, \cdots c_m\}$$

Formula 1

$$C = A \cap B = \{\ c_1, c_2, \cdots c_m\}$$

Formula 2

When $A = \{dog\ \}$ and $B = \{female\}$

$$A \cup B = \{all\ female\ animal\ includeing\ ladies\ and\ all\ dogs\}$$

$$A \cap B = \{female\ dog\}$$

From formula 1 and formula 2, $A \cap B$ is obviously subset of $A \cup B$, and generally

$$A \cup B \supseteq A \cap B$$

In the case

$$A = B,$$
$$A \cup B = A \cap B$$

Phenomenon B happened after happening of A, or on the condition of happening of B, it is expressed as $B|A$.

Generally

$$A \cap B = B \cap A$$

Though

$$(A|B) \neq (B|A)$$

When we express probability of phenomenon A as $P(A)$

In the case A and B happen independently.

$$P(A \cup B) = P(A) + P(B)$$

Formula 3

$$P(A \cap B) = P(A)P(B)$$

Formula 4

$$P(A|B) = P(B)P(A|B) = P(A)P(B)$$

Formula 5

Combination is a subset of which elements are selected from a set depending on a condition. When we roll a dice, and we consider pips of one is phenomenon A and the other number is phenomenon B, phenomenon A and phenomenon B is exclusive each other in a roll.

When we roll dice once, there are two cases. One is A and the other is B.

The number of each case is as follows.

Case 1 $\quad N(A) = 1$

Case 2 $\quad N(B) = 1$

Number of total cases is

$$N(A) + N(B) = 1 + 1 = 2$$

Probability of case A and probability of case B are as follows.

$$P(A) = \frac{1}{6}, \qquad P(B) = 1 - \frac{1}{6} = \frac{5}{6}$$

$$\because \; P(A \cup B) = 1$$

Where,

$N(A)$: number of case A, $\quad P(A)$: *possiblity of A*

$N(B)$: number of case B, $\quad P(B)$: *possiblity of B*

When we roll dice several times, possible case and the possibility of the cases can be simulated as follows.

Times of roll=2

Possible cases are A|A, B|A, A|B and B|B

Case 1 $\mathrm{N}(A|A) = 1$

Case 2 $\mathrm{N}(B|A) = 1$

Case 3 $\mathrm{N}(A|B) = 1$

Case 4 $\mathrm{N}(B|B) = 1$

Number of total cases is

$$\mathrm{N}(A|A) + \mathrm{N}(B|A) + N(A|B) + N(B|B) = 1 + 1 + 1 + 1 = 4$$

Probabilities of each case are

Case 1 $\mathrm{P}(A|A) = \frac{1}{6}\cdot\frac{1}{6}$

Case 2 $\mathrm{P}(B|A) = \frac{1}{6}\cdot\frac{5}{6}$

Case 3 $\mathrm{P}(A|B) = \frac{5}{6}\cdot\frac{1}{6}$

Case 4 $\mathrm{P}(B|B) = \frac{5}{6}\cdot\frac{5}{6}$

Among them, phenomenon A appears 2 times in case 1, 1 times in case 2 and 3, and 0 time in case 4.

When we consider possibility of times of appearance of phenomenon of A.

$$p(2) = \mathrm{P}(A|A) = \frac{1}{6}\cdot\frac{1}{6} = \frac{1}{36}$$

$$p(1) = \mathrm{P}(B|A) + \mathrm{P}(A|B) = \frac{1}{6}\cdot\frac{5}{6} + \frac{5}{6}\cdot\frac{1}{6} = 2\cdot\frac{1}{6}\cdot\frac{5}{6} = 2\cdot\frac{1}{6}\cdot\left(1-\frac{1}{6}\right) = \frac{10}{36}$$

$$p(0) = \mathrm{P}(B|B) = \frac{5}{6}\cdot\frac{5}{6} = \frac{25}{36}$$

Where

$p(i)$: possiblity of i times appearance in repeated trials.

When we throw dice three times (estimation of probability of times of pip of 1),

		times of phenomenon A	Possibility
Case 1	A | A | A	3	$\mathrm{P(A\|A\|A)} = \frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}$
Case 2	B | A | A	2	$\mathrm{P(B\|A\|A)} = \frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}$
Case 3	A | B | A	2	$\mathrm{P(A\|B\|A)} = \frac{1}{6}\times\frac{5}{6}\times\frac{1}{6}$
Case 4	A | A | B	2	$\mathrm{P(A\|A\|B)} = \frac{5}{6}\times\frac{1}{6}\times\frac{1}{6}$
Case 5	B | B | A	1	$\mathrm{P(B\|B\|A)} = \frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}$
Case 6	B | A | B	1	$\mathrm{P(B\|A\|B)} = \frac{5}{6}\times\frac{1}{6}\times\frac{5}{6}$
Case 7	A | B | B	1	$\mathrm{P(A\|B\|B)} = \frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}$

Case 8	B | B | B	0	$P(B\|B\|B) = \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}$

$$p(3) = P(A|A|A\,) = 1 \cdot \left(\frac{1}{6}\right)^3 = \frac{1}{216},$$

$$p(2) = P(B|A|A) + P(A|B|A) + P(A|A|B) = 3 \cdot \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right) = \frac{15}{216},$$

$$p(1) = P(B|B|A) + P(B|A|B) + P(A|B|B) = 3 \cdot \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^2 = \frac{75}{216},$$

$$P(0) = P(B|B|B) = 1 \cdot \left(\frac{5}{6}\right)^3 = \frac{125}{216}$$

$$p(3) + p(2) + p(1) + p(0) = 1$$

When we throw dice four times (estimation of probability of times of pip of 1),

		times of phenomenon A	Possibility
Case 1	A | A | A | A	4	$P(A\|A\|A\|A) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}$
Case 2	B | A | A | A	3	$P(B\|A\|A\|A) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6}$
Case 3	A | B | A | A	3	$P(A\|B\|A\|A) = \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6}$
Case 4	A | A | B | A	3	$P(A\|A\|B\|A) = \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6} \times \frac{1}{6}$
Case 5	A | A | A | B	3	$P(A\|A\|A\|B) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6}$
Case 6	B | B | A | A	2	$P(B\|B\|A\|A) = \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6}$
Case 7	B | A | B | A	2	$P(B\|A\|B\|A) = \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6}$
Case 8	A | B | B | A	2	$P(A\|B\|B\|A) = \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$
Case 9	B | A | A | B	2	$P(B\|A\|A\|B) = \frac{5}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6}$
Case 10	A | B | A | B	2	$P(B\|A\|A\|A) = \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6}$
Case 11	A | A | B | B	2	$P(A\|A\|B\|B) = \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} \times \frac{1}{6}$
Case 12	B | B | B | A	1	$P(B\|B\|B\|A) = \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}$
Case 13	B | B | A | B	1	$P(B\|B\|A\|B) = \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6}$
Case 14	B | A | B | B	1	$P(B\|A\|B\|B) = \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6}$
Case 15	A | B | B | B	1	$P(A\|B\|B\|B) = \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$
Case 16	B | B | B | B	0	$P(B\|B\|B\|B) = \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}$

$$p(4) = P(A|A|A\,|A) = 1 \cdot \left(\frac{1}{6}\right)^4 = \frac{1}{1296},$$

$$p(3) = P(B|A|A|A) + P(A|B|A|A) + P(A|A|B|A) + P(A|A|A|B) = 4 \cdot \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^1 = \frac{20}{1296},$$

$$p(2) = P(B|B|A|A) + P(B|A|B|A) + P(A|B|B|A) + P(B|A|A|B) + P(A|B|A|B) + P(A|A|B|B)$$

$$= 6 \cdot \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2 = \frac{150}{1296},$$

$$p(1) = \mathrm{P(B|B|B|A)} + \mathrm{P(B|B|A|B)} + \mathrm{P(B|A|B|B)} + \mathrm{P(A|B|B|B)} = 4 \cdot \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^3 = \frac{500}{1296},$$

$$P(0) = \mathrm{P(B|B|B|B)} = 1 \cdot \left(\frac{5}{6}\right)^4 = \frac{625}{1296}$$

$$p(4) + p(3) + p(2) + p(1) + p(0) = \frac{1 + 20 + 150 + 500 + 625}{1296} = 1$$

From these examples, we can derive a general formula to express possibility of times of a phenomenon (k) in a number of trials (n) as follows.

$$\mathrm{p(k)} = \mathrm{C} \cdot \mathrm{P(A)}^k \cdot (1 - \mathrm{P(A)})^{n-k}$$

Then we express C in above formula using *n* and *k*.

When we list up the all the cases of *k*=3, *n*=4

B|A|A|A, A|B|A|A, A|A|B|A, A|A|A|B

From this we can conclude that

$$\mathrm{C} = 4$$

This is the same as mathematics o combination and permutation such as following question.

" When there are three As and each A can select for positions namely position 1 to 4. How many possible case do exist?"

There are 4 positions as follow

1|2|3|4

First A (A_1) can select 4 positions

Second A (A_2) can select 3 positions except the position selected by A_1

Third A (A_3) can select 2 positions except the positions selected by A_1 and A_2

There are $4 \times 3 \times 2 = 24$ possible cases if we consider difference among A_1, A_2 and A_3. However, we don't need to consider the difference among A_1, A_2 and A_3.When we delete the open position, there remain a permutation of A_1, A_2 and A_3. And all possible cases include the permutation. That means we do not need to consider difference of A_1, A_2 A_3, A_1 A_2 A_3, A_2 A_1 A_3, A_2 A_3 A_1, A_3 A_1 A_2 and A_3 A_2 A_1. The possible number of the permutation can be calculated as follow.

$${}_3\mathrm{P}_3 = 3 \times 2 \times 1 = 6$$

There are 4 possible open positions and each open position equally contains 6 permutations composed from A_1, A_2 and A_3.

So, number of combination composed from $\mathrm{A, A, A}$ and not A is calculated as follow

$$\frac{4\times 3\times 2}{3\times 2x1}=\frac{24}{6}=4$$

There are several notations of combination taking *k* elements from *n* elements, such as ${}_{n}C_{k},\ C(n,k),\ {}^{n}C_{k},\ C_{n}^{k}$.

The author leaned the notation of ${}_{n}C_{k}$, in high schools in Japan. Using this notation, number of combination can be expressed as follow

$${}_{n}C_{k}=\frac{n\cdot(n-1)\cdot\cdots\big(n-(n-k)\big)}{k\cdot(k-1)\cdot\cdots\cdot 1}=\frac{n!}{k!\,(n-k)!}$$

This is expressed as follow in USA.

$$\binom{n}{k}=\frac{n!}{k!\,(n-k)!}$$

Formula 6

Using ${}_{n}C_{k}$, we can generally express the $p(k)$ as follow

$$p(k)=\ {}_{n}C_{k}p^{k}(1-p)^{n-k}$$

Or

$$p(k)=\binom{n}{k}p^{k}(1-p)^{n-k}$$

Formula 7

In the above case,

n=1

$$p(1)=\frac{1}{1}\left(\frac{1}{6}\right)^{1}\left(\frac{5}{6}\right)^{0}=\frac{1}{6},\qquad p(0)=\frac{1}{1}\left(\frac{1}{6}\right)^{0}\left(\frac{5}{6}\right)^{1}=\frac{5}{6}$$

n=2

$$p(2)=\frac{2\times 1}{2\times 1}\left(\frac{1}{6}\right)^{2}\left(\frac{5}{6}\right)^{0}=\frac{1}{36},p(1)=\frac{2}{1}\left(\frac{1}{6}\right)^{1}\left(\frac{5}{6}\right)^{1}=\frac{10}{36}\ \ p(0)=\frac{0}{0}\left(\frac{1}{6}\right)^{0}\left(\frac{5}{6}\right)^{2}=\frac{25}{6}$$

$$where\ \frac{0}{0}=1$$

n=3

$$p(3)=\frac{3\times 2\times 1}{3\times 2\times 1}\left(\frac{1}{6}\right)^{3}\left(\frac{5}{6}\right)^{0}=\frac{1}{216},p(2)=\frac{3\times 2}{2\times 1}\left(\frac{1}{6}\right)^{2}\left(\frac{5}{6}\right)^{1}=\frac{15}{216}\ \ p(1)=\frac{3}{1}\left(\frac{1}{6}\right)^{1}\left(\frac{5}{6}\right)^{2}=\frac{75}{216}$$

$$p(0)=\frac{0}{0}\left(\frac{1}{6}\right)^{0}\left(\frac{5}{6}\right)^{2}=\frac{125}{216}$$

n=4

$$p(4)=\frac{4\times 3\times 2\times 1}{4\times 3\times 2\times 1}\left(\frac{1}{6}\right)^{4}\left(\frac{5}{6}\right)^{0}=\frac{1}{1296},p(3)=\frac{4\times 3\times 2}{3\times 2\times 1}\left(\frac{1}{6}\right)^{3}\left(\frac{5}{6}\right)^{1}=\frac{20}{1296}$$

$$p(2)=\frac{4\times3}{2\times1}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2=\frac{150}{1296}, p(1)=\frac{4}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^3=\frac{500}{1296}, \quad p(0)=\frac{0}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^4=\frac{625}{1296}$$

n=5

$$p(5)=\frac{5\times4\times3\times2\times1}{5\times4\times3\times2\times1}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^0=\frac{1}{7776}, p(4)=\frac{5\times4\times3\times2}{4\times3\times2\times1}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^1=\frac{25}{7776}$$

$$p(3)=\frac{5\times4\times3}{3\times2\times1}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2=\frac{250}{7776}, p(2)=\frac{5\times4}{2\times1}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3=\frac{1250}{7776},$$

$$p(1)=\frac{5}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^4=\frac{3125}{7776}, p(0)=\frac{0}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^5=\frac{3125}{7776}$$

In above examples the author explained ${}_nC_r$ as number of combination. ${}_nC_r$ can explained as binomial coefficients.

Binomial coefficients is coefficients of each member of expansion of power of two terms.

and expressed as $\binom{n}{k}$ in general notation system in USA

$$(p+q)^n=\binom{n}{n}p^nq^0+\binom{n}{n-1}p^{n-1}q^1+\cdots+\binom{n}{n-k}p^{n-k}q^k+\cdots+\binom{n}{0}p^0q^n$$
$$=\sum_{k=0}^{n}\binom{n}{n-k}p^{n-k}q^k$$

Formula 8

When we line up binomial coefficients in a triangle shape, we can obtain Pascal's triangle is

$$\binom{0}{0}$$

$$\binom{1}{1}\quad\binom{1}{0}$$

$$\binom{2}{2}\quad\binom{2}{1}\quad\binom{2}{0}$$

$$\binom{3}{3}\quad\binom{3}{2}\quad\binom{3}{1}\quad\binom{3}{0}$$

$$\binom{4}{4}\quad\binom{4}{3}\quad\binom{4}{2}\quad\binom{4}{1}\quad\binom{4}{0}$$

$$\binom{5}{5}\quad\binom{5}{4}\quad\binom{5}{3}\quad\binom{5}{2}\quad\binom{5}{1}\quad\binom{5}{0}$$

The result of calculation of binomial coefficients

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

This triangle shows symmetric structure of binomial coefficients

$$\binom{n}{k} = \binom{n}{n-k}$$

and the other calculation method using recurrence relation

$$\binom{n}{k} = \binom{n-k}{k-1} + \binom{n-k}{k}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

$2 = 1 + 1$

$3 = 1 + 2$

$4 = 1 + 3$, $6 = 3 + 3$

When the possibility of times of happening of a phenomenon *p*(*k*) (*k* is times of happening of a phenomenon) in *n* trials and the possibility of happening of the phenomenon in a trial is *p* and *k*, we say *p*(k) follows binomial distribution model. This relation can be expressed as follow.

$$p(k) \sim B(n, p)$$

The following graph is showing $p(k) \sim B\left(n, \frac{1}{6}\right)$ with k ($1 \leq n \leq 10$)

We can understand that the shape of the distribution of the possibility became symmetric and unimodal distribution like normal distribution with increase of n.

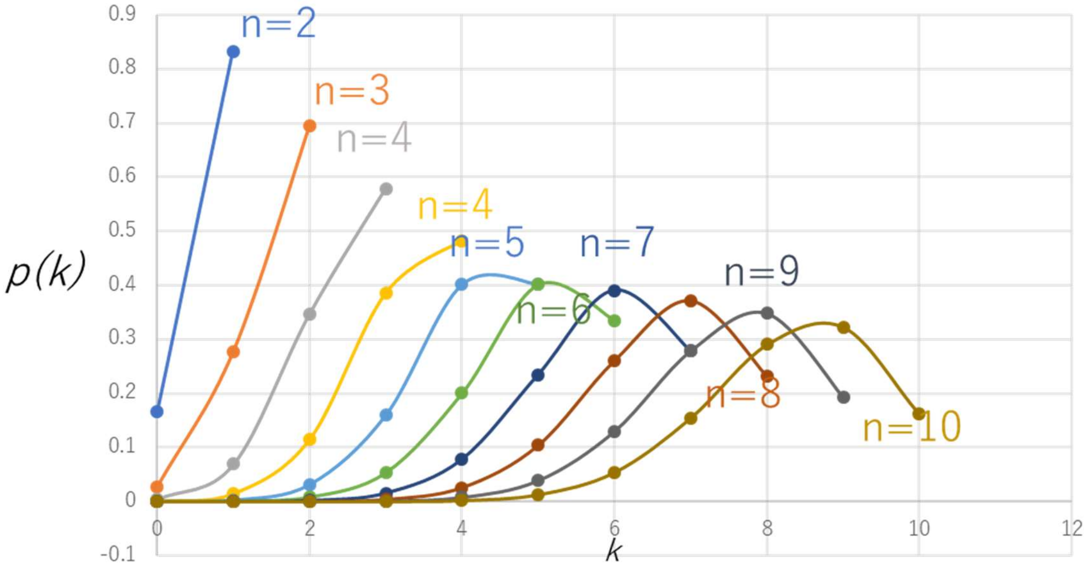

Fig. 5. Binomial distribution