III-2-3. Poisson distribution

Poisson distribution is obtained from binomial distribution by expanding n to infinity fixing expected value (np). As the result, probability (p) decrease to infinitesimal. Poisson distribution is used for statistical analysis of rare phenomena. The process to make Poisson distribution from binomial distribution is composed only from transformation of formula, though we should use definition of Napier's constant in the process. Readers who have no knowledge of Napier's constant, refer the paragraph of Napier's constant (III-3-2)

In Poisson distribution, $np = \mu\,(constant)$ we consider probability of a phenomena happening k times. Using binominal distribution

$$P(k) =_n C_k p^k q^{(n-k)}$$

$$= \frac{n!}{k!(n-k)!} p^k q^{(n-k)}$$

$$= \frac{n!}{k!\,(n-k)!} p^k (1-p)^{(n-k)}$$

$$\because p + q = 1$$

$$= \frac{1}{k!} \frac{n!}{(n-k)!} p^k (1-p)^{(n-k)}$$

We separate the formula to three part

$$= \frac{1}{k!} \mathrm{A} \cdot \mathrm{B}$$

$$\mathrm{A} = \frac{n!}{(n-k)!} p^k$$

$$\mathrm{B} = (1-p)^{(n-k)}$$

About A,

$$\mathrm{A} = \frac{n!}{(n-k)!} p^k$$

$$= \mathrm{n}(n-1) \cdots (n-k+1) p^k$$

$$= \frac{n}{n} \left(\frac{n-1}{n}\right) \cdots \left(\frac{n-k+1}{n}\right) n^k p^k$$

$$= 1 \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) (np)^k$$

$$Here, \qquad np = \mu$$

$$\mathrm{A} = 1 \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) (\mu)^k$$

When $n \to \infty$,

$$\frac{1}{n} \to 0, \frac{k-1}{n} \to 0$$

$$\lim_{n\to\infty} A = \mu^k$$

About B,

$$\mathrm{B} = (1-p)^{(n-k)}$$

Here, $np = \mu$.

$$\mathrm{B}=\left(1-\frac{\mu}{n}\right)^{\mu\frac{n}{\mu}-k}$$

$$= \frac{\left(1-\frac{\mu}{n}\right)^{\mu\frac{n}{\mu}}}{\left(1-\frac{\mu}{n}\right)^{-k}}$$

$When\, n \to \infty,$

$$\left(1-\frac{\mu}{n}\right) \to 1 \quad \left(1-\frac{\mu}{n}\right)^{-k} \to 1$$

Definition of Napier's constant

$$e = \lim_{\mathrm{n}\to\infty}\left(1+\frac{1}{\mathrm{n}}\right)^{\mathrm{n}}$$

$$\left(1-\frac{\mu}{n}\right)^{\mu\frac{n}{\mu}} = \left(\left(1-\frac{\mu}{n}\right)^{\frac{n}{\mu}}\right)^{\mu} = e^{-\mu}$$

$$\lim_{n\to\infty} \mathrm{B} = \frac{\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^{\mu\frac{n}{\mu}}}{\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^{-k}}$$

$$= \frac{e^{-\mu}}{1}$$

$$= e^{-\mu}$$

$$\lim_{n\to\infty} P(k) = \lim_{n\to\infty} {}_nC_k p^k q^{(1-k)}$$

$$= \lim_{n\to\infty} \frac{1}{k!}\frac{n!}{(n-k)!} p^k (1-p)^{(n-k)}$$

$$= \frac{1}{k!} \lim_{n\to\infty} \frac{n!}{(n-k)!} p^k \lim_{n\to\infty} (1-p)^{(n-k)}$$

$$= \frac{1}{k!} \lim_{n\to\infty} A \lim_{n\to\infty} B$$

$$= \frac{\mu^k e^{-\mu}}{k!}$$

We cannot know μ. We use mean value of date (λ) instead of μ

$$\mathrm{P(X = k)} = \frac{\lambda^k e^{-\lambda}}{k!}$$

Equation 16

Meaning of the equation is probability happening k times in a event.

In Poisson distribution variance changes depending on average. This is same as binominal distribution. The dependence of variance on mean is logically guessable, when the value of np is fixed and n is increased with decrease of p. When n increase, the peak of the distribution sharpens around the mean. From these fact, in perfect Poisson distribution, value of variance is the same as mean. This is important to determine whether the distribution can be treated as Poisson distribution or not.

Proof.

Simplified calculation of variance

$$V_{(k)} = E_{(k^2)} - {E_{(k)}}^2$$

$\mathrm{E}_{(k)}$ is np, from the date we use average λ .

$$\mathrm{P(X = k)} = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$E_{(k^2)} = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k e^{\lambda}}{k!}$$

$$= \sum_{k=0}^{\infty} k \frac{\lambda^k e^{\lambda}}{(k-1)!}$$

$$= \sum_{k=0}^{\infty} \frac{\{(k-1)+1\}\lambda^k e^{\lambda}}{(k-1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(k-1)\lambda^k e^{\lambda}}{(k-1)!} + \sum_{k=0}^{\infty} \frac{\lambda^k e^{\lambda}}{(k-1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(k-1)\lambda^k e^{\lambda}}{(k-1)!} + \lambda \sum_{k=0}^{\infty} \frac{\lambda^{(k-1)} e^{\lambda}}{(k-1)!}$$

$\sum_{k=0}^{\infty} \frac{\lambda^{(k-1)} e^{\lambda}}{(k-1)!}$ is sum of probability when k=k-1, So,

$$\sum_{k=0}^{\infty} \frac{\lambda^{(k-1)} e^{\lambda}}{(k-1)!} = 1$$

$$= \sum_{k=0}^{\infty} \frac{(k-1)\lambda^k e^{\lambda}}{(k-1)!} + \lambda$$

Repeat same procedure,

$$= \sum_{k=0}^{\infty} \frac{(k-1)\lambda^k e^{\lambda}}{(k-1)!} + \lambda$$

$$= \sum_{k=0}^{\infty} \frac{\lambda^k e^{\lambda}}{(k-2)!} + \lambda$$

$$= \lambda^2 \sum_{k=0}^{\infty} \frac{\lambda^{(k-2)} e^{\lambda}}{(k-2)!} + \lambda$$

$\sum_{k=0}^{\infty} \frac{\lambda^{(k-2)} e^{\lambda}}{(k-2)!}$ is sum of probability when k=k-2.

$$\sum_{k=0}^{\infty} \frac{\lambda^{(k-2)} e^{\lambda}}{(k-2)!} = 1$$

$$E_{(k^2)} = \lambda^2 + \lambda$$

$$E_{(k)} = \lambda$$

$$E_{(k)}{}^2 = \lambda^2$$

$$V_{(k)} = E_{(k^2)} - E_{(k)}{}^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$= \lambda$$

$$V_{(k)} = \lambda$$

Q.E.D