## III-3. Mathematical explanations

### III-3-1. Taylor expansion

Taylor expansion is a method to approximately transform complicate function to polynomial.  Coefficients of terms of polynomial are calculated from high-order derivatives and Taylor expansion is generally explained repeats of partial integration. However, target readers of this text are not familiar with mathematics. The author try to explain Taylor expansion by law of mean and concept of derivation.

III-3-1-1. law of mean and continuity

Definition of derivation of function is as follow

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f'(x)$ : derivate of $f(x)$

Formula 26

$f'(x)$ can be expressed as $\frac{df(x)}{dx}$ in another notation system. This notation is defining that the function should be derivate by $x$. Second order derivation is expressed as $f''(x)$ or $\frac{d^2f(x)}{dx^2}$. Higher order derivation such as n-order derivation is expressed as $f^{(n)}(x)$ or $\frac{d^nf(x)}{dx^n}$.

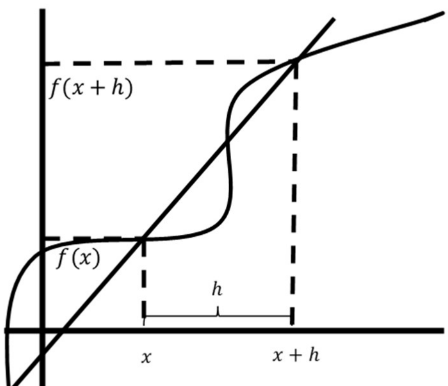


Fig.22. Concept of derivate

Curving line in figure 22 is function $f(x)$. We fix a point on the line. The point is $(x, f(x))$. Another point on the line is $(x+h, f(x+h))$. We consider a straight line passing both points. When we move the point $(x+h, f(x+h))$ on the curving line by changing h, the slope changes. The slope can be calculated as follow.

$$\frac{f(x+h)-f(x)}{h}$$

When we move the point to the nearest of $(x, f(x))$, if the slope reaches a definite value, we can descript the value as derivate value.

This is expressed by following formula.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

In another word, derivate value is slope of the tangent line at $(x, f(x))$.

When the line is continuous at $(x, f(x))$, we can also express the formula as follows.

$$f'(\mathrm{x}) = \lim_{h\to 0}\frac{f(x)-f(x-h)}{h}$$

In analytical mathematics, "continuous" is very important idea and we need cumbersome discussion using $\varepsilon-\delta$ logic for complete understanding. However, honestly, the author hates such discussions from my first lecture of mathematics in general education course nearly 50 years ago. The readers can understand Taylor expansion without such a boring lecture. Here "continuous" simply means the line does not reach $\infty$ or $-\infty$ or connect with other function at the point.

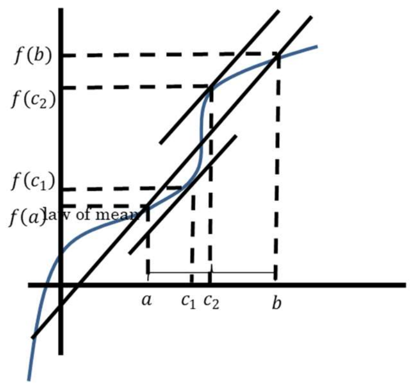


Fig.23. Concept of law of mean

III-3-1-2. Explanation of Taylor expansion by law of mean

Then we consider the relation between

$$\mathrm{f}'(\mathrm{x}) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

And

$$\frac{f(x+h)-f(x)}{h}$$

When $x=a,\ \ x+h=b,$

$$\frac{f(\ b)-f(a)}{b-a}$$

As shown in figure 23, above formula is slope of the line connecting $(a, f(a))$ and $(b, f(b))$. From this, we can understand that there are more than one point between $(a, f(a))$ and $(b, f(b))$at which the slope ( derivate at the point) is the same with the slope of the connecting line $f(x)$ in any shapes of line. In figure 23, $(c_1, f(c_1))$ and $(c_2, f(c_2))$ are such points $(a \le c \le b$.

This is expressed as follow.

There exist at least one c following the next formula

$$\frac{f(\ b)-f(a)}{b-a}=f'(c),\ \ (a \le c \le b)$$

Formula 27

The readers may consider this sentence tells own story, though this law is important as base of various propositions and formula.

What we want to do by Taylor expansion is approximation of function $f(x)$ of which shape is unknown because of some reason by data of a point$(a, f(a))$ and derivatives $f'(a), f''(a),\ \cdots\cdots f^{(n)}(a)$.

From law of mean,

$$\frac{f(x)-f(a)}{x-a}=f'(c_1),\quad \mathrm{a} \le c_1 \le \mathrm{x}$$

$$f(x)=f(a)+(x-a)f'(c_1)$$

Formula 28

As of now, we don't know $f(x)$. It is not likely to estimate $f'(c_1)$ and $c_1$.

If we know $a$, f$(a)$ and $f'(a)$, we can consider to use $f'(a)$ instead of $f'(c_1)$ as follow.

$$f(x) \doteqdot f(a)+(x-a)f'(a)$$

Of course, this is ridiculous. Generally, this is impossible. However, is this always impossible?

How about the case of $f(x)=x$ ?

$f'(x)=1$

$f'(x)$ is constant regardless of $x$

$$f'(x)=f'(c_1)=f'(a)=1$$

So in this case, we can say

$$f(x) = f(a) + (x - a)f'(a)$$

Confirmation

$$f(a) = a \text{ and } f'(a) = 1$$

We can put this in formula 28

$$f(x) = a + (x - a) * 1 = x$$

This relation is true in the case of $f(x) = \mathrm{B}x$ and $f(x) = \mathrm{A} + \mathrm{B}x$, because $f'(x)$ is constant.

The author supposes that several readers are angry, when they read above explanation. Because the line of the $f(x)$ is linear and the slope of constant in the case of $f(x) = \mathrm{B}x$ and $f(x) = \mathrm{A} + \mathrm{B}x$. There is no surprise.

How about in the case of $f(x) = x^2$?

$$f'(x) = 2x$$

$$f'(c_1) = 2c_1$$

$$f'(a) = 2a$$

So

$$f'(c_1) \neq f(a)$$

and we cannot put those in formula 28.

However, we do not need to give up here. We can use second order derivative. Before that, it is better to confirm the relation among $c_1$ $x$ and $a$.

From

$$\frac{f(x) - f(a)}{x - a} = f'(c_1)$$

, the relation is as follow.

$$\frac{x^2 - a^2}{x - a} = 2c_1$$

$$x + a = 2c_1$$

$$c_1 = \frac{x + a}{2}$$

Conclusively we can say that $c_1$ is average of $x$ and $a$ or that $c_1$ is at midpoint of $x$ and $a$.

Then the derivative is

$$f''(x) = 2$$

So,

$$f''(x) = f''(c_2) = f''(a) = 2$$

$c_2$ can be any point between $a$ and $x$.
So, using law of mean,

$$f'(x) = f'(a) + (x - a)f''(c_2)$$

then

$$f'(c_1) = f'(a) + (c_1 - a)f''(a)$$

The relation among $x$, $a$ and $a$ is

$$f(x) = f(a) + (x - a)f'(c_1)$$

So,

$$f(x) = f(a) + (x - a)\{f'(a) + (c_1 - a)f''(a)\}$$

Using $c_1 = \frac{x+a}{2}$,

$$f(x) = f(a) + (x - a)\left\{f'(a) + ((\tfrac{x+a}{2}) - a))f''(a)\right\}$$

$$f(x) = f(a) + (x - a)\left\{f'(a) + (\frac{x - a}{2})f''(a)\right\}$$

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x - a)^2}{2}f''(a)$$

Formula 29

Conformation in the case of $f(x) = x^2$

$$f(a) = a^2$$
$$f'(a) = 2a$$
$$f''(a) = 2$$

Right side members of formula 29

$$a^2 + (x - a)(2a) + \frac{(x - a)^2}{2} \times 2$$
$$= a^2 + 2ax - 2a^2 + x^2 - 2ax + a^2$$
$$= x^2$$

We could confirm $f(x) = x^2$
This relation is true in the case of $f(x) = \mathrm{A} + \mathrm{B}x + \mathrm{C}x^2$. The author thinks that we do not need confirmation by calculating value of derivatives. Simply we can understand by thinking $f(x) = \mathrm{A} + \mathrm{B}x + \mathrm{C}x^2$ is a result of parallel translation of $f(x) = \mathrm{C}x^2$.

The author thinks that most of readers already can suppose what the author will do in the next step.
Yes, let's try $f(x) = x^3$

$$f(x) = x^3$$
$$f'(x) = 3x^2$$

$$f''(x) = 3 * 2x = 6x$$
$$f'''(x) = 3 * 2 = 6$$
$$f(a) = a^3$$
$$f'(a) = 3a^2$$
$$f''(a) = 3 * 2a = 6a$$
$$f'''(a) = 3 * 2 = 6$$

We can obtain

$$f''(c_2) = f''(a) + (c_2 - a)f'''(a)$$

Then we have to put those results to $f(x) = f(a) + (x - a)f'(c_1)$ to get the conclusion, and obtainable conclusion is $f(x) = f(a) + \frac{(x-a)}{1}f'(a) + \frac{(x-a)^2}{2*1}f''(a) + \frac{(x-a)^3}{3*2*1}f'''(a)$.

However, the process of transformation is not sophisticated and elegant. We need long line to show the process, though reader can understand what we want to do by Taylor expansion. We can proof more simply by partial integration. The author agrees the opinion that the method using partial integration is more elegant. However, the reader should know higher skills of differentiation and integration for understanding the proof process using differentiation or integration.

III-3-1-3. Proof of Taylor expansion by derivatives of compound function

Differentiation of a compound function is a method of calculation of derivatives of compound functions.

The rule is expressed as follow.

When $z = f(y), y = g(x)$, and y is differentiable at $x_0$ $and$ $y$ is differentiable at $y_0$, where $y = g(x)$,

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

Proof

From the conditions following derivatives are exist

$$y' = \frac{dy}{dx} = \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}$$

$$z' = \frac{dz}{dy} = \lim_{y \to y_0} \frac{f(y) - f(y_0)}{y - y_0}$$

When $x \to x_0$,

$$y \to y_0$$

$$\because\ g(x) \text{ is continuous at } x = x_0,$$

(Some readers may consider this is trivial, though this non-trivial in any case. There are many function which has discontinuous points, we cannot fix value of the function at discontinuous points, When the function is differentiable at a point, the function is continuous at the point. See below)

$$\lim_{x \to x_0} \left(f(x) - f(x_0)\right) = \lim_{x \to x_0} (x - x_0) \cdot \lim_{x - x_0} \frac{f(x) - f(x_0)}{(x - x_0)}$$

$$= 0 \cdot f(x)$$

$$= 0$$

So, when $x \to x_0$,

$$f(x) - f(x_0) = 0$$

$$f(x) = f(x_0)$$

This is the definition of continuity at $x_0$

When we consider $(x-a)^n$ is a compound function as follow

$$\mathrm{z} = f(y) = y^n \text{ and } y = g(x) = x - a$$

$$(x-a)^n = f(y)$$

$$\frac{df(y)}{dy} = ny^{n-1}$$

$$\frac{dg(x)}{dx} = 1$$

$$f'(x) = \{(x-a)^n\}' = \frac{df(y)}{dy}\frac{dg(x)}{dx} = ny^{n-1} = n(\mathrm{x} - \mathrm{a})^{n-1}$$

$$\{(x-a)\}' = 1$$

$$\{(x-a)^2\}' = 2(x-a)$$

$$\left\{\frac{(x-a)^{n+1}}{n+1} + C\right\}' = (x-a)^n$$

This means that indefinite integral of $(x-a)^n$ is $\frac{(x-a)^{n+1}}{n+1} + \mathrm{C}$. (C is constant)

From $f''(c_2) = f''(a) + (c_2 - a)f'''(a)$, we can obtain following equation of line of tangency of $f'(x)$ by changing $c_2$ by

$$f''(x) = f''(a) + (x-a)f'''(\mathrm{a})$$

$f'(x)$ is integral of this equation. When we take definite integral in the interval of $(x, a)$ both side members,

$$f'(x) - f'(a) = (x-a)f''(a) + \frac{(x-a)^2}{2}f'''(a)$$

$$f'(x) = f'(a) + (x-a)f''(a) + \frac{(x-a)^2}{2}f'''(a)$$

Again, when we integrate both sides,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2} f''(a) + \frac{(x-a)^3}{2*3} f'''(a)$$

Here,

$$f(a) = a^3$$
$$f'(a) = 3a^2$$
$$f''(a) = 6a$$
$$f'''(a) = 6$$

So,

$$f(x) = a^3 + (x-a)*3a^2 + \frac{(x-a)^2}{2} : 6a + \frac{(x-a)^3}{3*2} * 6$$

$$f(x) = a^3 + 3a^2x - 3a^3 + 3ax^2 - 6a^2x + 3a^3 + x^3 - 3ax^2 + 3a^2 - a^3\ We$$
$$f(x) = x^3$$

We can confirm that we can write $f(x) = x^3$ estimate cubic formula, when we know the values of $a, f(a), f'(a), f''(a), f'''(a)$.
More elegant expression of this relation is as follow.

$$f(x) = f(a) + \frac{(x-a)}{1} f'(a) + \frac{(x-\mathrm{a})^2}{2*1} f''(a) + \frac{(x-\mathrm{a})^3}{3*2*1} f'''(a)$$

From these experiences, we suppose a general theory that when derivative value becomes constant after n times of differentiation. The function $f(x)$ can be expressed as follow.

$$f(x) = f(a) + \frac{(x-a)}{1} f'(a) + \frac{(x-a)^2}{2*1} f''(a) + \frac{(x-a)^3}{3*2*1} f'''(a) \ + \qquad + \frac{(x-a)^n}{n!} f^{(n)}(a)$$

Using $\Sigma$, the equation can be write as follow.

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a)$$

Formula30

A new question is whether we can generalize this relation to all functions, because we cannot always obtain constant after finite times of differentiation.
This is clear when we consider $f(x) = \sin x$.

$$f'(x) = \cos x$$
$$f''(x) = -\sin x$$
$$f'''(x) = -\cos x$$
$$f''''(x) = \sin x$$

This is a rotation and we cannot obtain constant, and we cannot use formula 27.
However, in the case $x$ is existing in neighborhood of $a$, $0 < |x-a| \ll 1$

, and we can approximately neglect higher order terms. In another word, it can be said that even if the shape of the line is complicated curving line, neighborhood is approximately liner. Derivate of liner line is constant.
So, in the case $x$ is existing in neighborhood of $a$.

$$f(x) \fallingdotseq f(a) + \frac{(x-a)}{1} f'(a) + \frac{(x-a)^2}{2*1} f''(a) + \frac{(x-a)^3}{3*2*1} f'''(a) \quad + \qquad + \frac{(x-a)^n}{n!} f^{(n)}(a)$$

Formula 31

This is Taylor expansion. Meaning of Taylor expansion is that we can approximately express all functions using multi order derivative values.

Required times of differentiation is depending on distance of $x$ and $a$, and allowable error.
Here, we consider the last term in right members of formula 31.

$$\frac{(x-a)^n}{n!} f^{(n)}(a)$$

When the formula is not approximate formula, the term should be written as follow.

$$\frac{(x-a)^n}{n!} f^{(n)}(c)$$

$c$ fluctuate in range of $(a, x)$, and $\frac{(x-a)^n}{n!} f^{(n)}(c)$ fluctuate with value of $c$. in the range of maximum and minimum value of $f^{(n)}(x)$ in the range of $(a, x)$.

So, error of the approximate formula $f(x)$ is within the fluctuation of $\frac{(x-a)^n}{n!} f^{(n)}(x)$.

We named $\frac{(x-a)^n}{n!} f^{(n)}(x)$ as surplus term.

III-3-1-4. proof of Taylor expansion by partial integration.
Before proof, several readers may want to confirm partial integration.
Partial integration is inverse operation of differentiation of multiplicated function.
Differentiation of multiplicated function is as follows.
When a function is in following form

$$F(x) = f(x)g(x)$$

We say the function is multiplicated function

We consider calculation of $\frac{dF(x)}{dx}$. This is differentiation of multiplicated function

From definition of derivative

$$\frac{dF(x)}{dx} = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

When we express $\frac{dF(x)}{dx} = F'(\mathrm{x})$

$$F'(\mathrm{x}) = \lim_{h \to 0} \frac{g(x+h)f(x+h) - g(x)f(x)}{h}$$

$$= \lim_{h \to 0} \frac{g(x+h)f(x+h) - g(x+h)f(x) + g(x+h)f(x) - (g(x)f(x)}{h}$$

$$\because -g(x+h)f(x) + g(x+h)f(x) = 0$$

$$= \lim_{h \to 0} \frac{g(x+h)\{f(x+h) - f(x)\} + f(x)\{g(x+h) - (g(x)\}}{h}$$

$$= \lim_{h \to 0} g(x+h) \frac{\{f(x+h) - f(x)\}}{h} + \lim_{h \to 0} f(x) \frac{g(x+h) - g(x)}{h}$$

$$= g(x)f'(x) + g'(x)f(x)$$

Q.E.D

This rule is generally expressed more simply using abridged notation.

$$\{g(x)f(x)\}' = g(x)f'(x) + g'(x)f(x)$$

Formula 32

Rule of partial integration is obtained by integration of both side of formula 32.

$$\int \{g(x)f(x)\}'dx = \int g(x)f'(x)dx + \int g'(x)f(x)dx$$

$$g(x)f(\mathrm{x}) = \int g(x)f'(x)dx + \int g'(x)f(x)dx$$

$$\int g'(x)f(x)dx = g(x)f(x) - \int g(x)f'(x)dx$$

Formula 33 (partial integration)

This formula is used generally in text books as a rule of partial integration. The author is memorizing the rule in following formula.

$$g(x)f(x) = \int g'(x)f(x)dx + \int g(x)f'(x)dx$$

or following definite integration form.

$$[g(t)f(t)]_a^x = \int_a^x g'(t)f(t)dt + \int_a^x g(t)f'(t)dt$$

Because the form is symmetric and easy to memory, and this form is sometimes convenient.

When we consider partial integration of $\int_a^x \frac{(x-t)^k}{k!}\frac{d^{k+1}f(t)}{dt^{k+1}}dt$

Here,

$$g(t) = \frac{(x-t)^k}{k!}$$

$$h(t) = \frac{d^k f(t)}{dt^k} = f^{(k)}(t)$$

$$\frac{dg(t)}{dt} = \frac{(x-t)^{k-1}}{(k-1)!}$$

$$\frac{dh(t)}{dt} = \frac{d^{k+1}f(t)}{dt^{k+1}} = f^{(k+1)}(t)$$

$$\frac{d^k f(t)}{dt^k} : k\ order\ derivative\ of\ f(t),\ \ f^{(k)}(t)$$

$$\left[\frac{(x-t)^k}{k!}\frac{d^k f(t)}{dt^k}\right]_a^x = -\int_a^x \frac{(x-t)^{k-1}}{(k-1)!}\frac{d^k f(t)}{dt^k}dt + \int_a^x \frac{(x-t)^k}{k!}\frac{d^{k+1}f(t)}{dt^{k+1}}dt$$

$$Left\ side = \frac{(x-x)^k}{k!}\frac{d^k f(t)}{dt^k} - \frac{(x-a)^k}{k!}\frac{d^k f(a)}{dt^k} = -\frac{(x-a)^k}{k!}\frac{d^k f(a)}{dt^k} = -\frac{(x-a)^k}{k!}f^{(k)}(a)$$

and

$$-\frac{(x-a)^{k+1}}{(k+1)!}\frac{d^{k+1}f(a)}{dt^{k+1}} = -\int_a^x \frac{(x-t)^k}{k!}\frac{d^{k+1}f(t)}{dt^{k+1}}dt + \int_a^x \frac{(x-t)^{k+1}}{(k+1)!}\frac{d^{k+2}f(t)}{dt^{k+2}}$$

When $k = 0$

$$\int \frac{df(t)}{dt}dt = f(t) + C$$

$$\int_a^x \frac{df(t)}{dt}dt = [f(t)]_a^x = f(x) - f(a)$$

Conclusively, we can get following recurrence formula

$$f(x) - f(a) = \int_a^x \frac{df(t)}{dt}dt$$

$$-(x-a)\frac{df(a)}{dt} = -\int_a^x \frac{df(t)}{dt}dt + \int_a^x (x-t)\frac{d^2 f(t)}{dt^2}dt$$

$$-\frac{(x-a)^2}{2}\frac{d^2 f(a)}{dt^2} = -\int_a^x (x-t)\frac{d^2 f(t)}{dt^2}dt + \int_a^x \frac{(x-t)^2}{2}\frac{d^3 f(t)}{dt^3}dt$$

$$-\frac{(x-t)^3}{3\cdot 2}\frac{d^3 f(t)}{dt^3} = -\int_a^x \frac{(x-t)^2}{2}\frac{d^3 f(t)}{dt^3}dt + \int_a^x \frac{(x-t)^3}{3\cdot 2}\frac{d^4 f(t)}{dt^4}dt$$

$$\vdots$$

$$-\frac{(x-a)^k}{k!}\frac{d^k f(a)}{dt^k} = -\int_a^x \frac{(x-t)^{k-1}}{(k-1)!}\frac{d^k f(t)}{dt^k}dt + \int_a^x \frac{(x-t)^k}{k!}\frac{d^{k+1} f(t)}{dt^{k+1}}dt$$

$$-\frac{(x-a)^{k+1}}{(k+1)!}\frac{d^{k+1} f(a)}{dt^{k+1}} = -\int_a^x \frac{(x-t)^k}{k!}\frac{d^{k+1} f(t)}{dt^{k+1}}dt + \int_a^x \frac{(x-t)^{k+1}}{(k+1)!}\frac{d^{k+2} f(t)}{dt^{k+2}}$$

$$\vdots$$

$$-\frac{(x-a)^{n-1}}{(n-1)!}\frac{d^{n-1} f(a)}{dt^{n-1}} = -\int_a^x \frac{(x-t)^{n-2}}{(n-2)!}\frac{d^{n-1} f(t)}{dt^{n-1}}dt + \int_a^x \frac{(x-t)^{n-1}}{(n-1)!}\frac{d^n f(t)}{dt^n}dt$$

$$-\frac{(x-a)^n}{n!}\frac{d^n f(a)}{dt^n} = -\int_a^x \frac{(x-t)^{n-1}}{(n-1)!}\frac{d^{k+1} f(t)}{dt^{k+1}}dt + \int_a^x \frac{(x-t)^n}{n!}\frac{d^{n+1} f(t)}{dt^{n+1}}dt$$

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<u>Sum up all members of left side and right side</u>

$$f(x) - f(a) - \sum_{k=1}^{n}\frac{(x-a)^k}{k!}\frac{d^k f(a)}{dt^k} = \int_a^x \frac{(x-t)^n}{n!}\frac{d^{n+1} f(t)}{dt^{n+1}}dt$$

$$f(x) = f(a) + \sum_{k=1}^{n}\frac{(x-a)^k}{k!}\frac{d^k f(a)}{dt^k} + \int_a^x \frac{(x-t)^n}{n!}\frac{d^{n+1} f(t)}{dt^{n+1}}dt$$

Final member ($\int_a^x \frac{(x-t)^n}{n!}\frac{d^{n+1} f(t)}{dt^{n+1}}dt$) is integration form of surplus term.

When second order derivation is possible

$$\int g'(x)f'(x)dx = g(x)f'(x) - \int g(x)f''(x)dx$$

Definite integration between $(a, b)$ is

$$\int_a^b g'(x)f'(x)dx = [g(x)f'(x)]_a^b - \int_a^b g(x)f''(x)dx$$

When $g'(x) = 1$,

$$\int_a^b g'(x)f'(x)dx = \int_a^b f'(x)dx = [f(x)]_a^b = f(b) - f(a)$$

$$\because \int f'(x)dx = f(x)$$

We have to consider $g(x)$ of which derivative $g'(x) = 1$

When $g(x) = -(b-x)$,

$$g'(x) = 1$$

$$\int_a^b f'(x)dx = [g(x)f'(x)]_a^b - \int_a^b g(x)f''(x)dx$$

$$[f(x]_a^b = [-(b-x)f'(x)]_a^b + \int_a^b (b-x)f''(x)dx$$

$$f(b)-f(a) = -(b-b)f'(b) - \{(-(b-a)f'(\mathrm{a}))\} + \int_a^{\mathrm{b}} (b-x)f''(x)dx$$

$$f(b)-f(a) = (b-a)f'(\mathrm{a}) + \int_a^b (b-x)f''(x)dx$$

When we consider that $g'(x) = (b-x)$.

$$g(x) = \frac{-(b-x)^2}{2}$$

,and $\int_a^b (b-x)f''(x)dx$ is a definite partial integration of $g'(x)f(x)$

$$\int_a^b (b-x)f''(x)dx = \left[\frac{-(b-x)^2}{2}f''(x)\right]_a^b + \int_a^b \frac{(b-x)^2}{2}f'''(x)$$

$$= \frac{(b-a)^2}{2}f''(a) + \int_a^b \frac{(b-x)^2}{2}f'''(x)$$

When we assign this result to $f(\mathrm{b}) - f(\mathrm{a}) = (b-a)f'(a) + \int_a^b (b-x)f''(x)dx$ば

$$f(b)-f(a) = (b-a)f'(a) + \frac{(b-a)^2}{2}f''(a) + \int_a^b \frac{(b-x)^2}{2}f'''(x)dx$$

Repeating this procedure,

$f(b) - f(a) =$

$$(b-a)f'(a) + \frac{(b-a)^2}{2}f''(a) + \frac{(b-a)^3}{3*2}f'''(a) + \cdots\cdots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \int_a^b \frac{(b-x)^{n-1}}{(n-1)!}f^{(n)}(x)dx$$

In the case $f(x)$ can be derivate n times and $f^{(n+1)}(x) = 0$

$f^{(n)}(x)$ is constant

So. we can move out $f^{(n)}(x)$ from integration

$$\int_a^b \frac{(b-x)^{n-1}}{(n-1)!}f^{(n)}(x)dx = f^{(n)}(x)\int_a^b \frac{(b-x))^{n-1}}{(n-1)!}dx$$

and

$$\int_a^b \frac{(b-x))^{n-1}}{(n-1)!}dx = \left[\frac{(b-x)^n}{n!}\right]_a^b$$

$$= \frac{(b-b)^n}{n!} - \left\{-\frac{(b-a)^n}{n!}\right\}$$

$$= \frac{(b-a)^n}{n!}$$

Then $f^n(x)$ is a slope of liner line of $f^{n-1}(x)$, and $f^{(n)}(x)$ is constant.

So

$$f^{(n)}(x) = f^{(n)}(a)$$

$$f(b) - f(a)$$

$$= (b-a)f'(a) + \frac{(b-a)^2}{2}f''(a) + \frac{(b-a)^3}{3*2}f'''(a) + \cdots\cdots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(b-a)^n}{n!}f^{(n)}(a)$$

Our purpose is express $f(x)$ using multiple order derivatives. So, we change $b$ by $x$, and transposition $f(a)$ to right members.

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \frac{(x-a)^3}{3*2}f'''(a) + \cdots\cdots + \frac{(x-a)^n}{n!}f^{(n)}(a)$$

Several readers may think proof is enough by this step, though we should consider cases when $f^{(n+1)}(x) \neq 0$

We will discuss whether we can change last member in right side of following equation by $\frac{(b-a)^n}{n!}f^n(c)$

$$f(b) - f(a) =$$

$$(b-a)f'(a) + \frac{(b-a)^2}{2}f''(a) + \frac{(b-a)^3}{3*2}f'''(a) + \cdots\cdots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \int_a^b \frac{(b-x)^{n-1}}{(n-1)!}f^{(n)}(x)dx$$

We do not know $f^n(x)$ in last term of right side.

$$\int_a^b \frac{(b-x)^{n-1}}{(n-1)!}f^n(x)dx$$

When $f^n(x)$ is constant we can push out $f^n(x)$ from integration as $f^n(x) = f^n(a)$. However, value of $f^n(x)$ fluctuates with $x$. One possible idea is pushing out $f^n(x)$ from inside of integration as constant giving error range. It is possible, because, even $f^n(x)$ fluctuate, the value is limited in a small range depending on the distance between $a$ and $b$.

So, we can approximately express $f^n(x)$ as follow.

$$f^n(x) \doteqdot f^n(c)$$

The relation is as follow

$$m \leq f^n(c) \leq M$$

$$m: minimum\ value$$

$$M: maximum\ value$$

$$a \leq c \leq b$$

When we express the value of $x$, which gives minimum and maximum value of $f^n(x)$,

$x_m$ and $x_M$.

$$f^n(x_m) = m、f^n(x_M) = M$$

In the case when $x_m \leq x_M$

$$a \leq x_m \leq c \leq x_M \leq b$$

In the case when $x_M \leq x_m$

$$a \leq x_M \leq c \leq x_m \leq b$$

$$\int_a^b \frac{(b-x)^{n-1}}{(n-1)!} f^n(x)dx \fallingdotseq \int_a^b \frac{(b-x)^{n-1}}{(n-1)!} f^n(c)dx = f^n(c)\int_a^b \frac{(b-x)^{n-1}}{(n-1)!} dx = \frac{(b-x)^n}{n!} f^n(c)$$

$$f(b) - f(a)$$

$$= (b-a)f'(a) + \frac{(b-a)^2}{2} f''(a) + \frac{(b-a)^3}{3*2} f'''(a) + \cdots\cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)^n}{n!} f^n(c)$$

Originally, our purpose is to express expression $f(x)$. So we change $b$ by $x$ and transposition $f(a)$ in left side to right side.

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2} f''(a) + \frac{(x-a)^3}{3*2} f'''(a) + \cdots\cdots + \frac{(x-a)^n}{n!} f^n(c)$$

The last term is surplus term. Surplus term is expressing error of estimation and error is called residuals and Generally, symbol residual is R.

$$|\mathrm{R}| \leq \frac{(b-a)^n}{n!} M.$$