III-3-2. Napier' s constant

Many people use Napier's constant without awareness. In concentration dependent reaction in chemistry, population growth of organisms in biology, light absorption in physics are expressed by exponential functions, Napier's constant is used. As the result of uses in various in disciplines, there are various definitions and expressions

$$e = \lim_{\mathrm{n}\to\infty}\left(1+\frac{1}{\mathrm{n}}\right)^{\mathrm{n}}$$

Formula 34

This is commonly used expression of Napier's constant, though The author cannot suppose how to use Napier's constant from this expression. Tentatively, we use this formula as the definition of Napier's constant.

Napier's constant is often used in calculus. Firstly, the author explains Napier's constant in calculus.
Most simple and understandable example is following differentiation

$$\frac{d\log_a x}{dx}$$

We cannot solve this differentiation without Napier's constant, and the great mathematician, Euler showed importance of Napier's constant from this formula. (On as side note, Napier's constant sometimes called Euler's constant.)

$$\frac{d\log_a x}{dx}$$

$$= \lim_{\Delta h\to 0}\frac{\log_a(x+\Delta h)-\log_a x}{\Delta h}$$

$$= \lim_{\Delta h\to 0}\frac{1}{\Delta h}(\log_a(x+\Delta h)-\log_a x)$$

$$= \lim_{\Delta h\to 0}\frac{1}{\Delta h}\log_a\frac{x+\Delta h}{x}$$

$$= \lim_{\Delta h\to 0}\frac{1}{\Delta h}\log_a\left(1+\frac{\Delta h}{x}\right)$$

Here,

$$\frac{1}{n} = \frac{\Delta h}{x}$$

When , $\Delta \mathrm{h} \to 0$, then $n \to \infty$

$$= \lim_{n\to\infty}\frac{n}{x}\log_a\left(1+\frac{1}{n}\right)$$

x is not including n. We can push out x to the outside of symbol of limit.

$$=\frac{1}{x}\lim_{n\to\infty} n\log_a\left(1+\frac{1}{n}\right)$$

$$=\frac{1}{x}\lim_{n\to\infty} \log_a\left(1+\frac{1}{n}\right)^n$$

The author wants to put symbol of limit inside of symbol of logarithm. This a question whether logarithm of limit is the same as limit of logarithm.   In this case the value is continuous, and we can put the limitation in logarithm.

$$=\frac{1}{x}\log_a\lim_{n\to 0}\left(1+\frac{1}{n}\right)^n$$

Inside of logarithm is definition of Napier's constant.

$$=\frac{1}{x}\log_a e$$

When the base of logarithm is e, $\log_e e$=1.

$$=\frac{1}{x}$$

Conclusively,

$$\frac{d\log_e x}{dx}=\frac{1}{x}$$

Formula 35

This means that when we take Napier's constant as base of logarithm, we can obtain the derivative of logarithm functions easily. Formula 35 is commonly used formula of differentiation of logarithm. We use e as base of natural logarithm. For this reason, base of natural logarithm is also other name of Napier's constant, and we calculate general logarithm function after base conversion to Napier's constant.

Napier's constant is irrational number, and we cannot express the true value by decimal number or fraction.

Followings are example of approximate calculation of Napier's constant following the definition.

n=1 $\left(1+\frac{1}{n}\right)^n=\left(1+\frac{1}{1}\right)^1=2$

n=2 $\left(1+\frac{1}{2}\right)^2=\left(1+2(\frac{1}{2})+\left(\frac{1}{2}\right)^2\right)=2{,}25$

n=3 $\left(1+\frac{1}{3}\right)^3=\left(1+3\left(\frac{1}{3}\right)+3\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^3\right)=2.37037$

n=4 $\left(1+\frac{1}{4}\right)^4 = \left(1+4\left(\frac{1}{4}\right)+6\left(\frac{1}{4}\right)^2+4\left(\frac{1}{4}\right)^3+\left(\frac{1}{4}\right)^4\right) = 2.44156$

$$\vdots$$

Actually, there are various definition of Napier's constant. Napier's constant is required in various situation, and defined by various formula. Lately, human being knew those were similar. Napier's value is existing in the world of infinity and various definition became similar at the point of infinity.

Above explanation is from differentiation of logarithm. We can also consider Napier's constant from differentiation of exponent function.
We consider following function.

$$\frac{df(x)}{dx} = f(x)$$

Formula 36

The meaning of this function is function of which derivative function is the same as original function.

To avoid complicated notation, we use $f'(\mathrm{x})$ and $f''(x)$ as expression of first order and second order derivatives.
From the meaning of the function,

$$f(x) = f'(x)$$

When we repeat this,

$$f(x) = f'(x) = f''(x) = f'''(x) = \cdots = f^n(x)$$

We consider Taylor expansion of this function.

$$f(x) = f(\mathrm{a}) + (x-\mathrm{a})f'(\mathrm{a}) + \frac{(x-\mathrm{a})^2}{2}f''(\mathrm{a}) + \frac{(x-a)^3}{2*3}f'''(a) + \cdots$$

$$= \mathrm{f(a)} + \lim_{n\to\infty}\sum_{k=1}^{n}\frac{f^k}{k!}(x-a)^k$$

Here,

$$f(0) = f'(0) = f''(0) = f'''(0) = \cdots = f^n(0)$$

So,

$$f(1) = f(0) + \frac{f'(0)}{1} + \frac{f''(0)}{2} + \frac{f'''(0)}{2*3} + \cdots$$

$$f(1) = f(0) + \frac{f(0)}{1} + \frac{f(0)}{2} + \frac{f(0)}{2*3} + \cdots$$

$$f(1) = f(0)\left(1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{2*3} + \cdots\right)$$

$$f(1) = f(0)\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right)$$
$$= \mathrm{f}(0)\left(1 + \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k!}\right)$$

We introduce Napier's constant as follow

$$1 + \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k!} = e$$

Formula 37

This is another definition of Napier's constant. This formula is used for easy approximate calculation of Napier's constant.

Most of common text books, $\lim_{\mathrm{n}\to\infty}\left(1+\frac{1}{\mathrm{n}}\right)^{\mathrm{n}}$ is used as the definition of Napier's constant, and formula 37 is introduced as Taylor series of $f(x)=e^{x}$ for rapid approximate calculation of Napier's constant.

Back on topic, when following equation is true

$$f(1) = f(0)\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right)$$
$$1 + \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k!} = e$$

Then

$$f(1) = f(0)e$$

We implement Taylor expansion at $x = 2,\ a = 1$.

$$f(x) = f(\mathrm{a}) + \lim_{n\to\infty}\sum_{k=1}^{n}\frac{f^{k}}{k!}(x-a)^{k}$$

$$f(2) = f(1) + \lim_{n\to\infty}\sum_{k=1}^{n}\frac{f(1)}{k!}(2-1)^{k}$$

$\because$ from $f(x) = f'(x)$

$$f(1) = f'(1) = f''(1) = f'''(1) = \cdots = f^{n}(1)$$

$$f(2) = f(1)\left(1 + \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k!}\right)$$

Here, $1 + \lim_{n\to 0}\sum_{k=1}^{n}\frac{1}{k!} = e$

$$f(2) = \mathrm{f}(1)e$$
$$= \mathrm{f}(0)\mathrm{e}^2$$

We repeat same procedure

$$f(3) = \mathrm{f}(0)\mathrm{e}^3$$
$$f(x) = \mathrm{f}(0)\mathrm{e}^x$$
$$f(0) = \mathrm{e}^0 = 1$$

Conclusively,

$$f(x) = e^x$$

We could show that the function $\frac{df(x)}{dx} = f(x)$ is $f(x) = e^x$。

Here, following question is remained. We know e and relation between logarithm and exponential as prior knowledge. Through the relation, we can accept the sameness.
This not proof.
The question is direct proof of

$$\lim_{\mathrm{n}\to\infty}\left(1 + \frac{1}{\mathrm{n}}\right)^{\mathrm{n}} = 1 + \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k!}$$

The author has tried and it was possible though the logic is somehow circular answer and not clear. The author is asking provision of clear solution using ideas with resourcefulness.