III-3-3. Jacobian

Jacobian is a concept which appears coordinate transformation. The concept itself is interesting for the author, though the explanation of the concept is difficult. For deeper understanding, readers need basic knowledge of matrix and vector. The author is recommending readers to read this paragraph again after understanding of multiple classification analysis.

There are various sizes. Length is a size in 1-dimension, area is 2-dimension (planar dimension), volumes is 3-dimension, the author does not know how tow say though there are 4-dimension, 5-dimension size and so on. Generally, sizes are expressed by multiplying orthogonal independent variables in different dimension, such as volume in 3 dimensions, length x width x height. When we express the size multiplying orthogonal (independent) elements in a coordinate, the volume is different when we see the figures from other coordinates. Jacobian is something like expansion rate with coordinate transformation. This is analogy. The expansion rate changes nature of size such as direction of integration and range of domain. One easy allegory is shadow graph. When we project a figure on a plane from a point of light source to another plane, the figure is enlarged and deformed. The relation of two figures on different plane is complicated depending on the angle of two plane. Moreover, when the planes have curves, the shape will be complicatedly deformed. This is a kind of transformation. For the author, meaning of transformation is a projection of map in a multidimensional vector space to the other multidimensional vector space.

Simultaneous equation is also a kind of projection. In Japan, students of elementary school lean so called “crane and turtle calculation procedure”.

Example of crane and turtle calculation procedure

Question

There cranes and turtles. Total number of individuals of cranes and turtle is 12, and total number of legs of crane and turtle are 38. Answer the number of cranes and turtle.

Solution

When all individuals are turtle, total number of legs should be $4 \times 12 = 48$. Actually, the total number of legs are 38. The difference is $48 - 38 = 10$. When we replace a turtle to a crane number of legs decrease 2. So, number of cranes is $10 \div 2 = 5$. And, the number of turtles is $12 - 5 = 7$

Answer

Crane: 5, Turtle: 7

Honestly speaking, this question includes no reality. No one thinks that there is a person who count the number of legs instead of counting number of individuals of cranes and number of turtles directly. However, this is formal beauty of Japanese culture. In Japanese culture, people believe that crane can live for 1,000 years and life span of turtle is 10,000years, and crane and turtle is classical subject of traditional picture as a lucky charm for long life. Some earlier educator considered this question and give the name of "crane and turtle calculation procedure".

The author tries to explain this question introducing idea of projection.
The relation between individual number of turtle and number of their legs is as follow

$$4I_t = l_t$$

I_t: number of individuals of turtle, l_t:number of legs of turtle

The author knows, it is a stretch to say 4 in the formula is Jacobian. However, the function is the magnification rate. This is similar to Jacobian. Meaning of the equation is that when we project a number on the line of I_t to line of l_t, the magnification rate is 4.

Similarly, the relation between individual number of crane and number of their legs is as follow

$$2I_c = l_c$$

I_c: number of individuals of crane, l_c:number of legs of turtle

Furthermore, the author wants to say derivatives are something like partial magnification rate in the projection.

$$\frac{dl_t}{dI_t} = 4$$

$$\frac{d\mathrm{f}_c}{dI_c} = 2$$

It is not surprising that the number are constant, because the function is liner and derivative of liner function coefficient is constant. We may not be able to say same thing in higher order functions and other functions. However, the neighborhood is liner even the curve of function is complicated. In another word, we can say we can estimate magnification rate by derivation and the rate depend on the angle between original space and transformed space around corresponding point. Of course, we are discussing in multidimensional space, we have to consider what is the angle between different spaces in corresponding points. However, if we segmentalize the space and consider around corresponding points, the authors strange imagination will have reality and we can definite the angle by Jacobian inversely.

Here, the author solves the question of crane and turtle calculation procedure using the solution method of simultaneous functions

$$u = f(x, y) = x + y$$

$$v = g(x, y) = 4x + 2y$$

$$4u = 4x + 4y$$

$$v = 4x + 2y$$

$$4u - v = 2y$$

$$y = \frac{4u - v}{2}$$

In the case, when $u = 12, v = 38$, $y = 5, x = 7$

Above process is generalized as follow

$$ax + by = u$$

$$cx + dy = v$$

Using the method of solution which learned in junior high school

$$acx + bcy = cu$$

$$acx + day = av$$

Deduction of lower equation form upper equation

$$bcy - ady = cu - av$$

$$y = \frac{cu - av}{bc - ad}$$

Change sing of numerator and dominator

$$y = \frac{av - cu}{ad - bc}$$

$$x = \frac{du - bv}{ad - bc}$$

Can you see this process is similar to shadow pictures.

$$x(ad - bc) = du - bv$$

$$y(ad - bc) = av - cu$$

When figure(x, y) on the $x - \mathrm{y}$ space is projected to $u - v$ space, the magnification rate is $(ad - bc)$ and map of $(du - bv, av - cu)$ is obtained.
Mathematically,

$$\mathrm{a}x + by = u$$

$$cx + dy = v$$

are conversion equation and $ad - bc$ is magnification rate by conversion.
When we write the simultaneous function by matrix operation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ \mathrm{y} \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

This is simple rewriting of simultaneous equation to matrix equation following the rule of matrix operation. In the expression, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is conversion and the conversion ratio (magnification rate) is expressed as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. This conversion ratio is named determinant. In this case, determinant is $ad - bc$. Determinant of 2×2 square matrix is calculated like this. Calculation of determinant is not the purpose of this paragraph, the reader need to know calculation of determinant, please refer text books of matrix operation.

As previously mentioned, the author is thinking that derivative is also magnification rate in segmentalized points. $\frac{\partial u}{\partial x}$ is a symbol of partial differentiation, that means differentiation of u only by x at (x, y) neglecting y. In another word, changes of u by smallness change of x at (x, y) neglecting y. The reason why we do not write as $\frac{du}{dx}$ is clarification of neglection of other variables.
However, briefly, when we derivate following equation by x

$$ax + by = u$$

obtainable derivative is

$$a = \frac{\partial u}{dx}$$

So,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ \mathrm{y} \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

can be rewrite as follow.

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \binom{x}{\mathrm{y}} = \binom{u}{v}$$

So, magnification rate is

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

This is Jacobian.

Several readers may claim that this is only the case of liner function and the adequacy of the explanation is still not confirmed. The author foresaw the criticism and prepare a justification that neighborhood is liner in all functions. Honestly, the author has no confidence about sufficiency of the justification. When we consider discontinuity of space, the justification may be not sufficient. The reason why the author selects the method of explanation by expansion of the logic from simultaneous functions is expectation of connection of matrix as notation of simultaneous functions to matrix as meaning of matrix as mapping and conversion.

General form of Jacobian is as follows.

Following is n order conversion.

$$x_i = \varphi_i(u_1, \cdots u_n) \qquad 1 \leq \mathrm{j} \leq \mathrm{n}$$

In this case,

$$\int \cdots \int_D f(x_1, \cdots x_n)\, dx_1 \cdots dx_n = \int \cdots \int_{D\prime} f\{\varphi_i(u_1, \cdots u_n), \cdots \varphi_n(u_1, \cdots u_n)\} \left[J_\varphi\right] du_1 \cdots du_2$$

(D is range of integration in $\boldsymbol{x}$: meaning that $f(x_1, \cdots x_n)$ is integrated in the range of $x_1, \cdots x_n$, D' is range of integration in $\boldsymbol{u}$)

$$\mathrm{J}_\varphi = \frac{d\boldsymbol{\varphi}}{d\boldsymbol{u}} = \begin{pmatrix} \frac{\partial \varphi_1}{\partial u_1} & \cdots & \frac{\partial \varphi_1}{\partial u_n} \\ \cdots & \ddots & \cdots \\ \frac{\partial \varphi_n}{\partial u_n} & \cdots & \frac{\partial \varphi_n}{\partial u_n} \end{pmatrix}$$

$$\left|\mathrm{J}_\varphi\right| = \begin{bmatrix} \frac{\partial \varphi_1}{\partial u_1} & \cdots & \frac{\partial \varphi_1}{\partial u_n} \\ \cdots & \ddots & \cdots \\ \frac{\partial \varphi_n}{\partial u_n} & \cdots & \frac{\partial \varphi_n}{\partial u_n} \end{bmatrix}$$

J_φ is Jacobian matrix, and $\left|\mathrm{J}_\varphi\right|$ is Jacobian.

As the author's explanation in this paragraph, we can calculate Jacobian without using Jacobian matrix. We used figures to understand space geometric relation. However, we cannot use such space geometric sense in higher dimension coordinate. Explanation using Jacobian matrix is understandable and acceptable in higher dimension coordinate. Jacobian matrix is necessary for generalization of coordinate conversion. The author will explain coordinate conversion using several examples in next chapter (III-3-4. Coordinate conversion). In the chapter author will show several example of calculation of Jacobian again.