III-3-4. Coordinate conversion

In the case when the direction of integration is not fit the direction of axes of the dimension in the coordinate, we cannot integrate the function easily. In such case, we try to transform the function to other coordinate. Polar coordinate transformation is often use in such case. A typical case is calculation of the surface area and volume of sphere. The Calculation can be expressed simply by multiple integration as follow.

$$I = \int\int\int_D dxdydz$$

$$\mathrm{D}: x^2 + y^2 + z^2 \leq r^2$$

When we try to calculate without transformation

$$I = \int_{-\sqrt{r^2-x^2-y^2}}^{\sqrt{r^2-x^2-y^2}} \int_{-\sqrt{r^2-x^2-z^2}}^{\sqrt{r^2-x^2-z^2}} \int_{-\sqrt{r^2-y^2-z^2}}^{\sqrt{r^2-y^2-z^2}} dxdydz$$

In this case, the directions of integration are fit the dimension of the coordinate, though we cannot remove any variance from the integration by each integration and finally we cannot obtain simple integration. However, there are several text books which introduce calculation procedure as if it can be possible only by multiple integral in orthogonal coordinate. Those procedure do not use polar coordinate transformation in written form of integration, though they also use polar coordinate transformation in the process. Following is typical example of such explanation

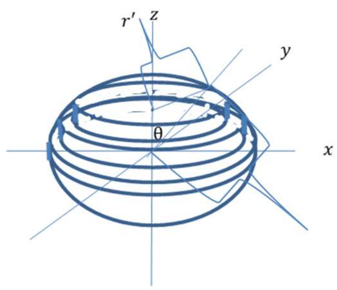


Fig. 24. Quadrature method of sphere in orthogonal coordinate.

We consider that sphere is a pile up of thin discs of different radius. When the radius of sphere is r, the radius of each disc r' is $r' = \sqrt{r^2 - z^2}$
and the area of the disc is

$$S = \pi r'^2 = \pi(r^2 - z^2)$$

The volume of the disc V_d is

$$\mathrm{V}_d = dzS = \pi(r^2 - z^2)dz$$

The volume of the sphere is obtainable by integration of the volume along z from -r to r

$$\int_{-r}^{r} Sdz = \int_{-r}^{r} \pi(r^2 - z^2)dz = \pi\left[r^2 z - \frac{z^3}{3}\right]_{-r}^{r} = \pi\left(\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right)\right) = \frac{4\pi r^3}{3}$$

Yes, this process does not include polar coordinate. However, this is not multiple integral, and they use the formula or area of circle (πr^2) as given knowledge. Conclusively, they use cylindrical polar coordinate. Cylindrical polar coordinate is a hybrid of orthogonal coordinate and polar coordinate. We can say that they used polar coordinate even partially. When we consider the formula of area of surface of sphere, we cannot use same method, because smallness width of circle line is not equable to the smallness z.

There are various polar coordinate depending on the dimension, and when we consider cylindrical polar coordinate as polar coordinate, the relation is complicated. However most sensuously understandable polar coordinate is sphere polar coordinate.

Transformation from orthogonal coordinate to sphere polar coordinate is as follows.

$$f(x, y, z) \rightarrow \boldsymbol{l}(l, \theta, \varphi)$$

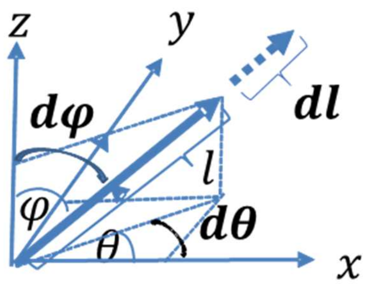


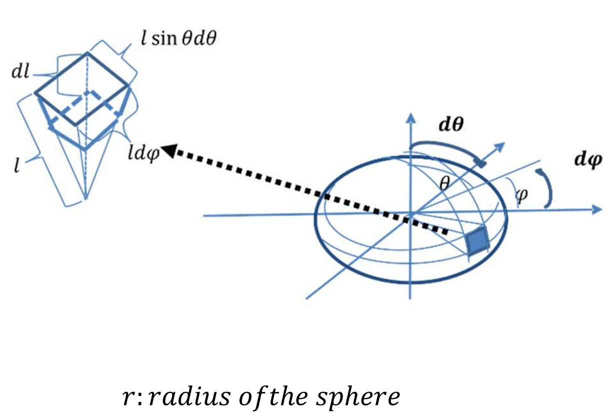
Fig.25. Relation of polar coordinate and orthogonal coordinate

Allows are indicating direction of angle

$$x = l \sin\theta \cos\varphi$$

$$y = l \sin\theta \sin\varphi$$

$$z = l \cos\theta$$



r: *radius of the sphere*

Fig. 26. Quadrature method of sphere in polar coordinate

When we consider the smallest unit in the polar coordinate, the smallest unit is expressed as the rectangular like solid inside of the heavy lines. Axes of r, θ, φ are independent and orthogonal each other. In this meaning we can say the box is orthogonal in the polar sphere coordination, though the axes are not orthogonal in orthogonal coordinate.

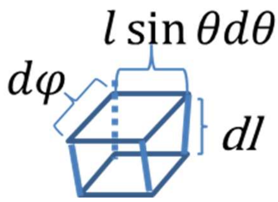


Fig. 27.　Unit of integration

The volume of the unit is

$$\mathrm{V}_u = l^2 \sin\theta dl d\theta d\varphi$$

The box is orthogonal in the polar coordination and we can integrate each direction independently.　We can obtain the surface area of sphere by multiple integration along $d\varphi$ from 0 to 2π and along θ from 0 to π.

$$S = \int_0^{2\pi}\int_0^{\pi} l^2 \sin\theta d\theta d\varphi = l^2\int_0^{2\pi} -[\cos\theta]_0^{\pi} d\varphi = l^2 \int_0^{2\pi} -(-1-1)d\varphi = 2l^2\int_0^{2\pi} -d\varphi$$

$$= 2l^2[\varphi]_0^{2\pi} = 4\pi l^2$$

$$l = r$$

$$S = 4\pi r^2$$

When we consider the sphere is tightly nested concentric sphere, we can obtain the volume by integration of the formula of surface area of sphere by integration along r from o to r.

$$V_s = \int_0^r 4\pi l^2 dl = 4\pi\int_0^r l^2 dl = 4\pi\left[\frac{l^3}{3}\right]_0^r = \frac{4}{3}\pi r^3$$

When we write the process of multiple integration from the beginning, the multiple integral can be expressed as follow.

$$V_s = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} l^2 \sin\theta \, dl d\theta d\varphi$$

$$\int_0^r l^2 dr = \frac{r^3}{3}$$

$$\int_0^{\pi} \sin\theta \, d\theta = 2$$

$$\int_0^{2\pi} d\varphi = 2\pi$$

$$V_s = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} l^2 \sin\theta \, dld\theta d\varphi = \int_0^{r} l^2 dl \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi = \frac{4}{3}\pi r^3$$

In the beginning of present paragraph, the author showed a process of formulation of area of volume of sphere in orthogonal coordinate and explain that the process was essentially multiple integration in cylindrical polar coordinate without showing the process of formulation in cylindrical polar coordinate.

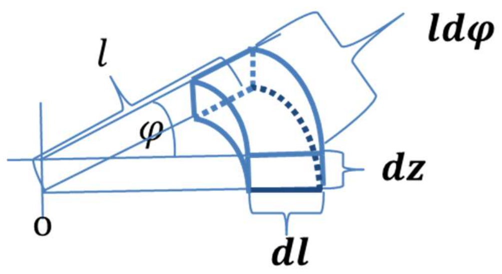


Fig.28. Unit of integration in Cylindrical coordination

The volume of the smallness unit of the volume in the coordinate is

$$V_u = l \, dldzd\varphi$$

When we calculate the area of the disc, we integrate the unit along r and φ.

$$S_d = \int_0^{2\pi}\int_0^{r} l \, dld\varphi = \int_0^{2\pi}\left[\frac{1}{2}l^2\right]_0^r d\varphi = \frac{1}{2}r^2\int_0^{2\pi} d\varphi = \frac{1}{2}r^2[\varphi]_0^{2\pi} = \pi r^2$$

When we calculate the volume of the sphere, we integrate the unit along r, φ and θ.

$$V_s = \int_0^{2\pi}\int_0^{r}\int_{-l}^{l} l \, dzdl \, d\varphi = \int_0^{2\pi}\int_0^{r} l[z]_{-l}^{l} \, drd\varphi = \int_0^{2\pi}\int_0^{r} 2l^2 dldz = 2\int_0^{2\pi}\left[\frac{1}{3}l^3\right]_0^r d\varphi$$

$$= \frac{2}{3}r^3\int_0^{2\pi} d\varphi = \frac{2}{3}r^3[\varphi]_0^{2\pi} = \frac{4\pi r^3}{3}$$

When we carefully look back the process, we can understand following facts.

In the case cylindrical polar coordinate, the smallest unit is expressed as follow.

$$V_u = l \, dldzd\varphi$$

In the case sphere polar coordinate, the smallest unit is expressed as follow.

$$V_u = l^2 \sin\theta dld\theta d\varphi$$

Both formula are composed from multiplication of all smallness unit of dimension and other part, such as l and $l^2 \sin\theta$. When we consider $drd\theta d\varphi$ and $drdzd\varphi$ as unit of the volume or size in the coordinate, l and $l^2 \sin\theta$ look like coefficients of size. We call such term as Jacobian. Function of the Jacobian is expansion rate between two coordinates. Of course, the expansion rates change with integration direction and integration range, so parts of Jacobian are also objective of integration. We can obtain Jacobian by drawing the shape of the smallness unit. However, in higher dimension, it is not always possible. Generally Jacobian is calculated as determinant of Jacob matrix, which is round robin partial differentiation.

Following is an example of Jacobian of transformation from orthogonal coordinate to sphere polar coordinate.

$$f(x, y, z) \rightarrow \boldsymbol{l}(l, \theta, \varphi)$$
$$x = l \sin\theta \cos\varphi$$
$$y = l \sin\theta \sin\varphi$$
$$\mathrm{z} = l \cos\theta$$

Jacobian matrix

$$\mathrm{J} = \begin{pmatrix} \frac{\partial x}{\partial l} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial l} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial l} & \frac{\partial z}{{}^\circ\mathrm{F}\theta} & \frac{\partial z}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & l\cos\theta\cos\varphi & -l\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & l\cos\theta\sin\varphi & l\sin\theta\cos\varphi \\ \cos\theta & -l\sin\theta & 0 \end{pmatrix}$$

Jacobian

$$|\mathrm{J}| = \begin{vmatrix} \sin\theta\cos\varphi & l\cos\theta\cos\varphi & l\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & l\cos\theta\sin\varphi & l\sin\theta\cos\varphi \\ \cos\theta & -l\sin\theta & 0 \end{vmatrix}$$

$$= 0 + l^2 \sin\theta \cos\theta^2 \cos^2\varphi + l^2 \sin^3\theta \sin^2\varphi + l^2 \sin^3\theta \cos^2\varphi + 0 + l^2 \sin\theta \cos^2\theta \sin^2\varphi$$
$$= l^2 \sin\theta \cos\theta^2 \cos^2\varphi + l^2 \sin\theta \cos^2\theta \sin^2\varphi + l^2 \sin^3\theta \sin^2\varphi + l^2 \sin^3\theta \cos^2\varphi$$
$$= l^2 \sin\theta \cos\theta^2 (\cos^2\varphi + \sin^2\varphi) + l^2 \sin^3\theta (\sin^2\varphi + \cos^2\varphi)$$
$$= l^2 \sin\theta (\sin^2\theta + \cos\theta^2)$$
$$= l^2 \sin\theta$$

So,

$$dxdydz = |J| dld\theta d\varphi = l^2 \sin\theta dld\theta d\varphi$$

and

$$\int\int\int_D dxdydz = \int\int\int_D l^2 \sin\theta dld\theta d\varphi$$

Similarly, an example of cylindrical polar coordinate is as follows.

Following is an example of Jacobian of transformation from orthogonal coordinate to cylindrical coordination.

$$f(x,y,z)\ \rightarrow \boldsymbol{l}(l,\varphi,z\,)$$

$$x = l\cos\varphi$$

$$y = l\sin\varphi$$

$$\mathrm{z} = z$$

Jacobian matrix

$$\mathrm{J} = \begin{pmatrix} \frac{\partial x}{\partial l} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial l} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial l} & \frac{\partial z}{°\mathrm{F}\theta} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\varphi & -l\sin\varphi & 0 \\ \sin\varphi & l\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Jacobian

$$|\mathrm{J}| = \begin{vmatrix} \cos\varphi & -l\sin\varphi & 0 \\ \sin\varphi & l\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= l\cos^2\varphi + l\sin^2\varphi$$

$$= l$$

So,

$$dxdydz = |J|dld\varphi dz = ldl\varphi dz$$

and

$$\int\int\int_D dxdydz = \int\int\int_D ldl\varphi dz$$

For the confirmation, the author picks up an example from this text book.
In the process of making normal distribution from binominal distribution, we make multiple integral from multiplying of two simple integrals. This is invers operation of operation making multiplying two simple integrals from multiple integral.

$$\int_0^\infty e^{-x^2}\,dx \times \int_0^\infty e^{-y^2}\,dy = \int_0^\infty\int_0^\infty e^{-x^2} \times e^{-y^2}\,dxdy$$

The purpose of this inverse operation is making polar coordinate. Originally x *and* y are the same. So, the shape of probability distribution is like a mountain of which contour lines are concentric. When we cut the mountain on the line include center, the cross sections are the same regardless the direction. When we can calculate the area of the cross section, we can obtain the volume of the mountain by integration along the circle. This is simply expressed in cylindrical polar coordinate.

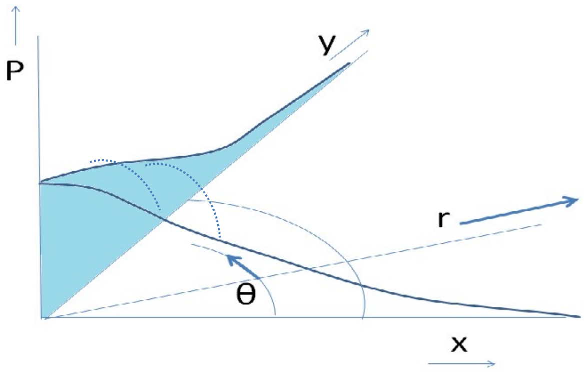


Fig. 29 Shape of distribution of probability

$$\int_0^\infty \int_0^\infty e^{-x^2} \times e^{-y^2}\, dxdy$$

$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dxdy$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\mathrm{z} = \mathrm{z}$$

Jacobian

$$|\mathrm{J}| = \begin{vmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dxdy = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\, rdrd\theta$$

$\because x^2 + y^2 \rightarrow\rightarrow r^2$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

$\int_0^\infty \quad \rightarrow\rightarrow \int_0^\infty$

When $x \to \infty$ and $y \to \infty$, then $x^2 + y^2 \to \infty$ *and* $r \to \infty$ $(r \geq 0)$

$\int_0^\infty \quad \to \to \int_0^{\frac{\pi}{2}}$

$$\tan\theta = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r}$$

When $y \to \infty$,

$$\tan\theta \to \infty \text{ and } \theta \to \frac{\pi}{2}$$

$$s = r^2$$

$$\frac{\partial s}{\partial r} = 2r$$

$$r\partial r = \frac{\partial s}{2}$$

When $r \to \infty$, then $s \to \infty$

$$\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r dr d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-s} ds d\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} [e^{-s}]_0^\infty d\theta = -\frac{1}{2}\int_0^{\frac{\pi}{2}} (0-1) d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{1}{2}[\theta]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

This is a beautiful example of application of polar coordinate.