III-3-5. Multiple integration

In statistics based on frequentism, ideal probability model is needed. Those models are produced by combining distribution models depending on nature of data and purpose of statistics. One typical example of combining process is making normal distribution from binominal distribution. Generally, those processes are composed from deformation of binominal distribution and calculation of probability distribution on multiple order plane. Several wise readers may thing that this explanation include antilogy linguistically, because plane means 2-dimensional space originally. However, here, author is thinking that n+1-dimensional space and value on the rank n+1 vector is determined by the values on the other vectors. When we consider the points in n order space which is expressed as $(x_1,\ x_2,\cdots x_n)$, the value on the rank n+1 vector is determined by the point as $\mathrm{P}(x_1,\ x_2,\cdots x_n)$. This is an analogy of the heights of the land which are determined by the points on 2-dimensional plane in maps. If we continue the analogy, the procedure to calculate the volume of the mountain is an integration of the height of the land in the region of the mountain in the plane. The name of this procedure is Multiple integration in mathematics, and multiple integration is expressed as follow.

$$\int\int\cdots\int_D P(x_1,x_2,\cdots x_n)dx_1dx_2\cdots dx_n$$

D is domain, the range of integration in n order space.

There are various cases of D.

In case when D is as follow,

$$\mathrm{D}=\{(x_1,x_2,\cdots x_n)|a\le x_1\le b,c\le x_2\le d,\ \ \cdots,y\le x_n\le z\}$$

We can express the multiple integration as follow

$$\int_y^z\cdots\int_c^d\int_a^b P(x_1,x_2,\cdots x_n)dx_1dx_2\cdots dx_n$$

However, in the case when domain is as follow,

$$\mathrm{D}=\{(x_1,x_2,\cdots x_n)|{x_1}^2+{x_2}^2+\cdots+{x_n}^2\le r^2\}$$

We cannot describe the region of each variance in each integration symbol, and we have to write the integration as follow

$$\int\int\cdots\int_D P(x_1,x_2,\cdots x_n)dx_1dx_2\cdots dx_n$$

$$\mathrm{D}=\{(x_1,x_2,\cdots x_n)|{x_1}^2+{x_2}^2+\cdots+{x_n}^2\le r^2\}$$

The simplest case of $P(x_1,x_2,\cdots x_n)$ is

$$P(x_1,x_2,\cdots x_n)=1$$

When we consider, $P(x,y,z)=1$

Its Multiple integration is

$$I = \iiint_D dxdydz$$

In the case $\mathrm{D} = \{(x, y, z) | a \leq x \leq b, c \leq y \leq d,\ \ e \leq z \leq f\}$

$$I = \int_e^f \int_c^d \int_a^b dxdydz$$

$$\int_b^a dx = [x]_a^b = b - a$$

$$I = \int_e^f \int_c^d (b-a)dydz = (b-a)\int_e^f \int_c^d dydz$$

$$\int_c^d dy = [y]_c^d = d - c$$

$$I = (b-a)\int_e^f (d-c)dz = (b-a)(d-c)\int_e^f dz$$

$$\int_e^f dz = [z]_e^f = f - e$$

$$I = (b-a)(d-c)\int_e^f dz = (b-a)(d-c)(f-e)$$

Meaning of the integration, $I = \int_e^f \int_c^d \int_a^b dxdydz$, is summing up the volume of all small cube, of which volume is expressed as $dxdydz$ or $\Delta x \Delta y \Delta z$, in the range. So, the meaning of the integration is accumulation of the volume of rectangular solid. (see figure 30).

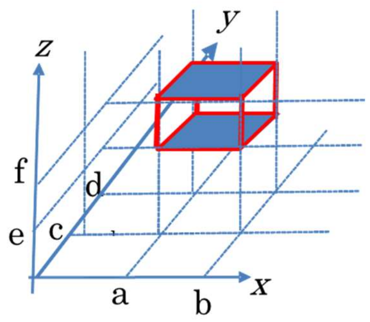

Figure 30. Multiple integration of $\int_e^f \int_c^d \int_a^b dxdydz$

Generally, multiple integration has meaning in spatial figure.

When $\mathrm{P}(x, y) = \frac{1}{2}x^2 - xy - x + 2y$,

The meaning of following multiple integration is volume of enclosed space by heavy lines in figure 31.

$$\int\int_{D} \mathrm{P}(x,y)dxdy$$

$$\mathrm{P}(x,y) = \frac{1}{2}x^2 - xy - x + 2y$$

$$\mathrm{D} = \left\{(x,y)|0 \le \mathrm{x} \le 2, 0 \le y \le \frac{1}{2}x\right\}$$

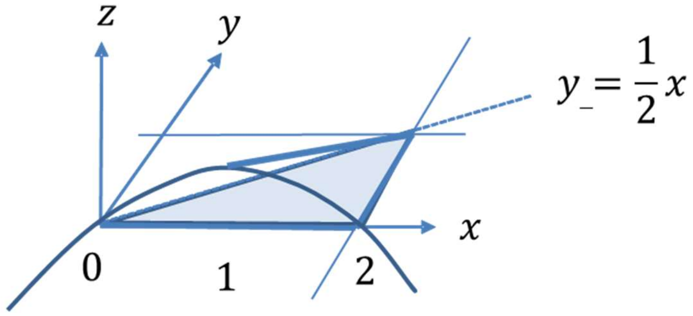

Figure 31. Multiple integration of $\int\int_{D} \mathrm{P}(x,y)dxdy$ $\mathrm{D} = \left\{(x,y)|0 \le \mathrm{x} \le 2, 0 \le y \le \frac{1}{2}x\right\}$

The meaning of the multiple integral is calculation of the volume of the 3 dimensional figure surrounded by heavy lines.

The example of calculation of the multiple integration is as follow

$$I = \int\int_{D} \left(\frac{1}{2}x^2 - xy - x + 2y\right) dydx$$

$$I = \int_0^2 \int_0^{\frac{1}{2}x} \left(\frac{1}{2}x^2 - xy - x + 2y\right) dydx$$

$$\int_0^{\frac{1}{2}x} \left(\frac{1}{2}x^2 - xy - x + 2y\right) dy = \left[\frac{1}{2}x^2y - \frac{1}{2}xy^2 - xy + y^2\right]_0^{\frac{1}{2}x} = \frac{1}{4}x^3 - \frac{1}{8}x^3 - \frac{1}{2}x^2 + \frac{1}{4}x^2$$

$$= x^2\left(\frac{1}{8}x - \frac{1}{4}\right)$$

$$I = \int_0^2 x^2\left(\frac{1}{8}x - \frac{1}{4}\right) dx = \left[\frac{1}{2^5}x^4 - \frac{1}{2^2 \cdot 3}x^2\right]_0^2 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

In this case, variable y could be removed by first integration and we could reach simple integral. In several cases, we could finally reach simple integral after repeated integral. However, we cannot always expect to directly reach simple integral from original multiple integral without transformation, because the domain often includes complicated functions. A typical case is calculation of the volume of sphere. The Calculation can be expressed simply by multiple integration as follow.

$$I = \int\int\int_D dxdydz$$

$$\mathrm{D} = \{(x, y.z)|x^2 + y^2 + z^2 \leq r^2\}$$

When we try to calculate without transformation

$$I = \int_{-\sqrt{r^2-x^2-y^2}}^{\sqrt{r^2-x^2-y^2}} \int_{-\sqrt{r^2-x^2-z^2}}^{\sqrt{r^2-x^2-z^2}} \int_{-\sqrt{r^2-y^2-z^2}}^{\sqrt{r^2-y^2-z^2}} dxdydz$$

Several variances remain in final form and we cannot transform the formula to single integration. In such case, we need to transform the original function to other form. In this this case, the domain is sphere, and most simple transformation is polar coordinate trans formation. Transformation to polar coordinate is often used for simplification of multiple integral, and the author wrote a paragraph for explanation of polar coordinate (III-3-4. Polar coordinate). However, polar coordinate is not only the method for simplification. The point is removal of variances from integrated function by the repeated integrals. Sometimes this is obtainable even by simple transformation from an orthogonal coordinate to another orthogonal coordinate by rotation of axes. In this text book, such orthogonal coordination to orthogonal coordinate transformation by rotation is used for accordance of direction of integration to dimensions of coordinate in formulization of χ^2 distribution. (Readers can obtain calculation procedure in III-2-5. χ^2 distribution).