IV-2-3. Structure of data

There are data sets which include plural factors. The author already pointed out that there are hidden variances which we cannot discuss by the method used in chapter IV-2-2, because the model used in the chapter is produced by combining two factors. Actual data include at least two more hidden factors other than factors demonstratively recognized. One is fluctuation caused by combination of recognized factor. This is named as interaction. The other is random fluctuation. This is named as residual. In the case of impact of fertilizer used in chapter IV-3, In a level of nitrogen, phosphate may have strong impact or little impact. Such impacts are not caused only simple effect of nitrogen and phosphate, but sometimes stronger impact is caused by particular combinations of nitrogen and phosphate. Sometimes no impact or inverse impact is caused by several particular combinations of nitrogen and phosphate. Those are examples of interaction. Distribution of organisms fluctuate by season and place, and particular places has particular functions in particular season. We need to detect such combination effect in ecological survey. Residual is fluctuation which we cannot be explained by factors which we can demonstratively recognize. It is difficult to explain what is residual, because we cannot know all back ground of phenomena. We interpret the fluctuation as random fluctuation, without consideration of possible mechanism causing the fluctuation. We can only say residual means the analyst recognize the fluctuation as residual. Firstly, we discuss interaction. We add small value to the values in each cell of the round robin table used in IV-2-2 in order to make interaction. Table 16 is the list of the value to add.

Table 16 data to add to each cell

	A_1	A_2	A_3	Sum	Mean
B_1	0.9	0.5	0.1	1.5	0.5
B_2	0.4	0.3	0.8	1.5	0.5
B_3	0.2	0.7	0.6	1.5	0.5
B_4	0.5	0.5	0.5	1.5	0.5
Sum	2.0	2.0	2.0	6.0	0.5
Mean	0.5	0.5	0.5	0.5	

In this table, averages in a line are the same among lines and averages in a column is the same among columns. This means that the average of line and column increase similarly. The addition of the values makes no change in the variance of line and column, though the total variance increases by variance of data to add. This is key point of this explanation.

We calculate SS of data to add

Table 17. Calculation of square

	A_1	A_2	A_3
B_1	$(0.9-0.5)^2$	$(0.5-0.5)^2$	$(0.1-0.5)^2$
B_2	$(0.4-0.5)^2$	$(0.3-0.5)^2$	$(0.8-0.5)^2$
B_3	$(0.2-0.5)^2$	$(0.7-0.5)^2$	$(0.6-0.5)^2$
B_4	$(0.5-0.5)^2$	$(0.5-0.5)^2$	$(0.5-0.5)^2$

Results

	A1	A2	A3	Sum
B1	0.16	0	0.16	0.32
B2	0.01	0.04	0.09	0.14
B3	0.09	0.04	0.01	0.14
B4	0	0	0	0.
Sum	0.26	0.08	0.26	0.60

SS_i =0.60

Then we add the data in table 16 to data in table 7 and combine two tables.

Table 18. Combined data

	A1	A2	A3	Sum	Mean
B1	2.9	6.5	7.1	16.5	5.5
B2	6.4	10.3	11.8	28.5	9.5
B3	7.2	11.7	12.6	31.5	10.5
B4	9.5	13.5	14.5	37.5	12.5
Sum	26	42	46	114	9.5
Mean	6.5	10.5	11.5	9.5	

Table 19. Calculation of SS

	A1	A2	A3	Sum
B1	43.56	9	5.76	58.32
B2	9.61	0.64	5.29	15.54
B3	5.29	4.84	9.61	19.74
B4	0	16	25	41.00
Sum	58.46	30.48	45.66	134.60

Total SS is 134.60

Most of readers may think this value as follow.

$$134.60 = 56 + 78 + 0.60 = 4 \times 14 + 3 \times 78 + 0.60 = n_B SS_A + n_A SS_B + SS_i$$

This is correct and an example of the theory that total SS is sum of partial SS, though this is only an empirical understanding. More logical explanation is as follows

The calculation of square of the distance from total average in cell A1B1is as follow.が。

$$(2.9 - 9.5)^2 = (6.6)^2 = 43.56$$

This is originally following calculation

$$\{(1-4) + (1-5) + (0.9-0.5)\}^2 = 43.56$$

Values written in red figure come from factor A, values written in blue figure come from factor B, and values written in green character come from interaction.

Expansion of the equation

$$(1-4)^2 + (1-5)^2 + (0.9-0.5)^2 + 2\{(1-4)(1-5) + (1-4)(0.9-0.5) + (1-5)(0.9-0.5)\}$$

In the line of B 1 , A2B1 cell

$$(5-4)^2 + (1-5)^2 + (0.5-0.5)^2 + 2\{(5-4)(1-5) + (5-4)(0.5-0.5) + (1-5)(0.5-0.5)\}$$

A3B1cell

$$(6-4)^2 + (1-5)^2 + (0.1-0.5)^2 + 2\{(6-4)(1-5) + (6-4)(0.1-0.5) + (1-5)(0.1-0.5)\}$$

Sum of line B1

$$(1-4)^2 + (5-4)^2 + (6-4)^2 + 3(1-5)^2 + (0.9-0.5)^2 + (0.5-0.5)^2 + (0.1-0.5)^2 + 2\{(1-5)((1-4) + (5-4) + (6-4)) + (1-4)(0.9-0.5) + (5-4)(0.5-0.5) + (6-4)(0.1-0.5) + (1-5)((0.9-0.5) + (0.5-0.5) + (0.1-0.5))\}$$

$$= (1-4)^2 + (5-4)^2 + (6-4)^2 + 3(1-5)^2 + (0.9-0.5)^2 + (0.5-0.5)^2 + (0.1-0.5)^2 + 2\{(1-4)(0.9-0.5) + (5-4)(0.5-0.5) + (6-4)(0.1-0.5)\}$$

$$= SS_{\hat{A}} + 3(1-5)^2 + (0.9-0.5)^2 + (0.5-0.5)^2 + (0.1-0.5)^2 + 2\{(1-4)(0.9-0.5) + (5-4)(0.5-0.5) + (6-4)(0.1-0.5)\}$$

$$\because\ (1-4) + (5-4) + (6-4) = 0$$

$$(0.9-0.5) + (0.5-0.5) + (0.1-0.5) = 0$$

Repeat similar procedure from B1 to B4 and sum up the result

$$SS_{total} = 4SS_{\hat{A}} + 3\{(1-5)^2 + (5-5)^2 + (6-5)^2 + (8-5)^2\}$$
$$+ \{(0.9-0.5)^2 + (0.5-0.5)^2 + (0.1-0.5)^2 + (0.4-0.5)^2 + (0.3-0.5)^2$$
$$+ (0.8-0.5)^2 + (0.2-0.5)^2 + (0.7-0.5)^2 + (0.6-0.5)^2 + (0.5-0.5)^2 + (0.5-0.5)^2$$
$$+ (0.5-0.5)^2\}$$
$$+2\{(1-4)((0.9-0.5) + (0.4-0.5) + (0.2-0.5) + (0.5-0.5)) + (5-4)((0.5-0.5) + (0.3-0.5) + (0.7-0.5) + (0.5-0.5)) + (6-4)((0.1-0.5) + (0.8-0.5) + (0.6-0.5) + (0.5-0.5))\}$$

$$(0.9 - 0.5) + (0.4 - 0.5) + (0.2 - 0.5) + (0.5 - 0.5) = 0$$
$$(0.5 - 0.5) + (0.3 - 0.5) + (0.7 - 0.5) + (0.5 - 0.5) = 0$$
$$(0.1 - 0.5) + (0.8 - 0.5) + (0.6 - 0.5) + (0.5 - 0.5) = 0$$
$$SS_{total} = 4SS_A + 3SS_B + SS_{interaction} = n_B SS_A + n_B SS_B + SS_{interaction}$$

This result means total SS can be decompose to partial SSs.

Then we need degree of freedom of each factor for the calculation of variances.

$$\text{Degree of freedom of factor A: } \mathrm{df}_A = 2$$
$$\text{Degree of freedom of factor B: } \mathrm{df}_B = 3$$
$$\text{Total degree of freedom: } \mathrm{df}_{total} = 11$$

When we apply the theory that total df is sum of partial df, we can calculate df_i as follow.

$$\mathrm{df}_{total} = \mathrm{df}_A + \mathrm{df}_B + df_i$$
$$df_{interaction} = \mathrm{df}_{total} - \mathrm{df}_A - \mathrm{df}_B = 11 - 2 - 3 = 6$$

This is correct, though we need to confirm existence of other factors. See Table 16. The table is dataset to make interaction. The table has 3 columns and 4 rows. Average of each line and column is the same. Each row includes 3 cells and degree of freedom in the line is 2. Each column includes 4 cells and degree of freedom in the column is 3. When we consider factor A and factor B is independent and orthogonal. The degree of freedom is

$$(3 - 1)(4 - 1) = 6$$

When we summarize the results more generalized expression

$$\text{total degree of freedom : } \mathrm{df}_{total} = n_A n_B - 1$$
$$\text{degree of freedom of factor A:: } \mathrm{df}_A = n_A - 1$$
$$\text{degree of freedom of factor B:: } \mathrm{df}_B = n_B - 1$$
$$\text{degree of freedom interaction : } \mathrm{df}_{interaction} = (n_A - 1)(n_B - 1)$$
$$\mathrm{df}_{total} = \mathrm{df}_A + \mathrm{df}_B + \mathrm{df}_i$$
$$n_A n_b - 1 = (n_A - 1) + (n_B - 1) + (n_A - 1)(n_B - 1)$$

Here, n_A is number of columns and n_B is number of lines.

Actually, we can obtain df_i by following two calculations

$$\mathrm{df}_{interaction} = (n_A - 1)(n_B - 1)$$
$$\mathrm{df}_{interaction} = \mathrm{df}_{total} - \mathrm{df}_A - \mathrm{df}_B$$

Generally , second calculation is commonly used because of easiness.

Conclusively,

$$n_B SS_A = 4 \times SS_A = 56$$
$$SS_A = 14$$

$$\sigma_A{}^2 = \frac{14}{2} = 7$$

$$n_B SS_B = 3 \times SS_B = 7$$

$$\sigma_B{}^2 \frac{SS_B}{df_B} = \frac{26}{3} = 8.6667$$

$$\sigma_{interaction}{}^2 = \frac{SS_{interaction}}{df_{interaction}} = \frac{0.6}{6} = 0.1$$

$$F_{A-interaction} = \frac{\sigma_A{}^2}{\sigma_{interaction}{}^2} = \frac{7}{0.1} = 70$$

$$F_{B-interaction} = \frac{\sigma_B{}^2}{\sigma_{interaction}{}^2} = \frac{8.6667}{0.1} = 86.667$$

We consider that added value to make interaction is caused only by interaction. However, we cannot say the fluctuation of the data is produced only by effect of interaction, it may include random fluctuation and impacts of minor factors. This means that we have to consider separation of interaction and residual. However, it is not possible in two-way ANOVA without repeats, because there is no data of unexplained data fluctuation. If there are repeat in a cell of same combination of factors, we can detect data fluctuation by unknown factor. and separate the such variance from interaction as residual. As usual, we make a model dataset which include repeats. The author makes three replications in a cell by adding values.

Table 20. List of value to add for making random fluctuation

	A_1	A_2	A_3
B_1	0.9	0.4	0.2
	0.5	0.3	0.7
	0.1	0.8	0.6
B_2	0.9	0.5	0.1
	0.4	0.3	0.8
	0.2	0.7	0.6
B_3	0.5	0.5	0.6
	0.4	0.5	0.3
	0.6	0.5	0.6
B_4	0.6	0.4	0.4
	0.4	0.4	0.2
	0.5	0.7	0.9

The average in cells are the same (0.5). This is for making random variance without any changes in other variances. We calculate the variance of added dataset by simplified calculation procedure. At first, explanation of calculation procedure. Following is transformation of definition of SS

$$\sum_i^n (x_i - \bar{x})^2 = \sum_i^n {x_i}^2 - 2\sum_i^n x_i\bar{x} + n\bar{x}^2$$

$\bar{x}$:average of x_i

$$\sum_i^n x_i\bar{x} = \bar{x}\sum_i^n x_i = \frac{\sum_i^n x_i}{n}\sum_i^n x_i = \frac{1}{n}\sum_i^n x_i^{\;2}$$

$$n\bar{x}^2 = n\left(\frac{\sum_i^n x_i}{n}\right)^2$$

$$\because \bar{x} = \frac{\sum_i^n x_i}{n}$$

Simplified calculation is as follow

$$SS = \sum_i^n (x_i - \bar{x})^2 = S - \frac{T^2}{n}$$

Calculation of T

				Sum
	0.9	0.4	0.2	1.5
	0.5	0.3	0.7	1.5
	0.1	0.8	0.6	1.5
	0.9	0.5	0.1	1.5
	0.4	0.3	0.8	1.5
	0.2	0.7	0.6	1.5
	0.5	0.5	0.6	1.6
	0.4	0.5	0.3	1.2
	0.6	0.5	0.6	1.7
	0.6	0.4	0.4	1.4
	0.4	0.4	0.2	1
	0.5	0.7	0.9	2.1
Sum	6	6	6	18

Calculation of S				Sum
	0.81	0.16	0.04	1.01
	0.25	0.09	0.49	0.83

	0.01	0.64	0.36	1.01
	0.81	0.25	0.01	1.07
	0.16	0.09	0.64	0.89
	0.04	0.49	0.36	0.89
	0.25	0.25	0.36	0.86
	0.16	0.25	0.09	0.5
	0.36	0.25	0.36	0.97
	0.36	0.16	0.16	0.68
	0.16	0.16	0.04	0.36
	0.25	0.49	0.81	1.55
Sum	3.62	3.28	3.72	10.62

$$\sum_{i}^{n} x_i = T$$

$$\sum_{i}^{n} {x_i}^2 = S$$

Using this transformation

$$\mathrm{SS} = \mathrm{S} - \frac{T^2}{n} = 10.62 - \frac{18^2}{36} = 1.62$$

In old days, we did not have computer, and we need to consider simplified calculation method. Today, we can use computer and knowledge of such calculation technique is not necessary. However, it may useful for understanding structure of variances.

Table 21. Combined dataset

	A_1	A_2	A_3
B_1	3.8	6.9	7.3
	3.4	6.8	7.8
	3.0	7.3	7.7
B_2	7.3	10.8	11.9
	6.8	10.6	12.6
	6.6	11.0	12.4
B_3	7.7	12.2	13.2
	7.6	12.2	12.9
	7.8	12.2	13.2
B_4	10.1	13.9	14.9
	9.9	13.9	14.7
	10.0	14.2	15.4

Using introduced calculation procedure, we calculate total SS.

Calculation of T

				Sum
	3.8	6.9	7.3	18
	3.4	6.8	7.8	18
	3	7.3	7.7	18
	7.3	10.8	11.9	30
	6.8	10.6	12.6	30
	6.6	11	12.4	30
	7.7	12.2	13.2	33.1
	7.6	12.2	12.9	32.7
	7.8	12.2	13.2	33.2
	10.1	13.9	14.9	38.9
	9.9	13.9	14.7	38.5
	10	14.2	15.4	39.6
Sum	84	132	144	360

Calculation of S

				Sum
	14.44	47.61	53.29	115.34
	11.56	46.24	60.84	118.64
	9	53.29	59.29	121.58
	53.29	116.64	141.61	311.54
	46.24	112.36	158.76	317.36
	43.56	121	153.76	318.32
	59.29	148.84	174.24	382.37
	57.76	148.84	166.41	373.01
	60.84	148.84	174.24	383.92
	102.01	193.21	222.01	517.23
	98.01	193.21	216.09	507.31
	100	201.64	237.16	538.8
Sum	656	1531.72	1817.7	4005.42

$$\mathrm{SS} = \mathrm{S} - \frac{T^2}{n} = 4005.42 - \frac{360^2}{36} = 405.42$$

This result is fit for our expectation that total SS is sum of partial SS

$$SS_{total} = n_{residual}(n_B SS_A + n_A SS_B + SS_{\widehat{interaction}}) + SS_{residual}$$
$$= 3(4 \times 14 + 3 \times 26 + 0.6) + 1.62 = 405.42$$

$$n_{residual}: number\ of\ replication = 3$$

Calculation of variances

$$\sigma_A{}^2 = \frac{SS_A}{\mathrm{df}_A} == \frac{14}{3-1} = 7$$

$$\sigma_B{}^2 = \frac{SS_B}{\mathrm{df}_B} = \frac{26}{4-1} = 8.6667$$

$$\sigma_{interaction}{}^2 = \frac{SS_{interaction}}{\mathrm{df}_{interaction}} = \frac{SS_{interaction}}{(n_A-1)(n_B-1)} = \frac{0.6}{2\times 3} = 0.1$$

$$\sigma_{residual}{}^2 = \frac{SS_{residual}}{\mathrm{df}_{residual}} = \frac{SS_{residual}}{(n_{residual}n_An_B-1)-\mathrm{df}_A-\mathrm{df}_B-\mathrm{df}_{interaction}} = \frac{1.62}{35-2-3-6}$$

$$= \frac{1.62}{24} = 0.0675$$

$$\sigma_{residual}: variance\ of\ residual$$

$$\sigma_{interaction}: variance\ of\ interaction。$$

Calculation of F ratio

$$F_{A-residual} = \frac{7}{0.0675} = 103.7037$$

$$F_{B-residua} = \frac{8.6667}{0.0675} = 118.5185$$

$$F_{interaction-residual} = \frac{0.1}{0.0675} = 1.481481$$

$$F_{A-interactio} = \frac{7}{0.1} = 70$$

$$F_{B-interacti} = \frac{8.6667}{0.1} = 86.667$$

From this result, we can conclude that the data spreads by factor A and B are enough large comparing to variance of residual and interaction. We can say that the data fluctuations are caused by main factor A and B. Existence of impact of interaction is not obvious, because the threshold value of F (p=0.05 $\mathrm{df}_{numerator} = 6$ and $\mathrm{df}_{doninator} = 2$) is 19.330

Above explanation is empirical by making dataset model and following the calculation process. The author is thinking that the explanation method is useful for understanding, though we need to add more theoretical explanation for deeper understanding. A method of logical explanation is standardization of data (to make average 0). This model is used previously in IV-2-2. We repeat same discussion again with more detailed explanation.

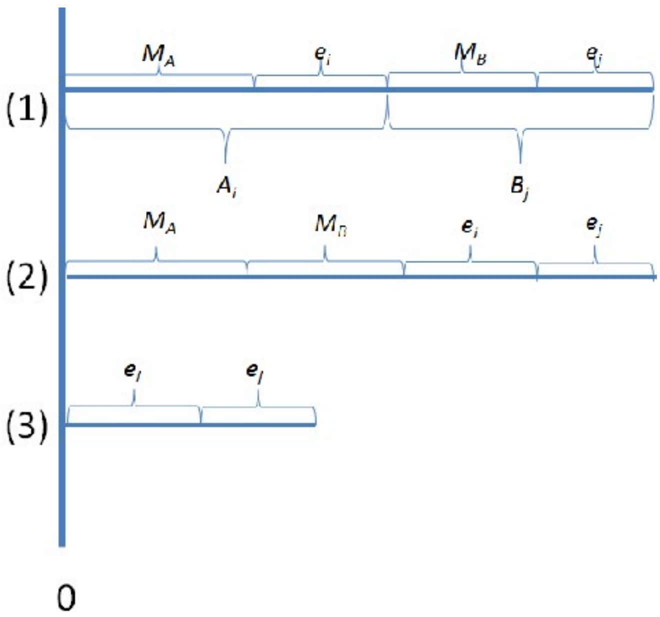

Fig. 32. Structure of data

When we subtract averages from the combined data (1), (3) is obtainable. We express the length of (3) as e_{A+B}, e_{A+B} is distance from the average, and standardized data by removing averages.

$$e_{A+B_{ij}} = e_{A_i} + e_{B_j}$$

When we express the average of e_{A_i} as $\bar{e_A}$.

$$\bar{e_A} = \frac{\sum_{i=1}^{n_A} e_{A_i}}{n_A}$$

Secondary moment of e_{A_i} (average of square of distance from) is as follow

$$\overline{e_A{}^2} = \frac{\sum_{i=1}^{n_A} e_{A_i}{}^2}{n_A}$$

$$\overline{e_A{}^2}: secondary\ moment$$

Average of all data is as follow.

$$\sum_{i=1}^{n_A}\sum_{j}^{n_B} e_{A+B_{ij}} = \sum_{i=1}^{n_A}\sum_{j}^{n_B}\left(e_{A_i} + e_{B_j}\right) = \sum_{i=1}^{n_A}\left(n_B e_{A_i} + \sum_{j}^{n_B} e_{B_j}\right) = n_B\sum_{i=1}^{n_A} e_{A_i} + n_A\sum_{j}^{n_B} e_{B_j} = 0$$

$$n_{total} = mn$$

Secondary moment of total dataset is

$$\overline{e_{A+B}{}^2} = \frac{SS_{A+B}}{n_A n_B} = \frac{\sum_{i=1}^{n_A}\sum_{j}^{n_B} e_{A+B_{ij}}{}^2}{n_A n_B} = \frac{\sum_{i=1}^{n_A}\sum_{j=1}^{n_B}\left(e_{A_i} + e_{B_j}\right)^2}{n_A n_B}$$

Expansion of $\overline{e_{A+B}{}^2} = \frac{SS_{A+B}}{n_A n_B}$

$$\frac{SS_{A+B}}{n_A n_B} = \frac{\sum_{i=1}^{n_A}\sum_{j=1}^{n_B}\left(e_{A_i} + e_{B_j}\right)^2}{n_A n_B} = \frac{1}{n_A n_B}\left\{\sum_{i=1}^{n_A}\sum_{j=1}^{n_B} e_{A_i}{}^2 + 2\sum_{i=1}^{n_A}\sum_{j=1}^{n_B} e_{A_i}e_{B_j} + \right\}$$

$$= \frac{1}{n_A n_B}\left\{n_B \sum_{i=1}^{n_A} e_{A_i}{}^2 + 2\sum_{i=1}^{n_A} e_{A_i}\sum_{j=1}^{n_B} e_{B_j} + n_A \sum_{j=1}^{n_B} e_{B_j}{}^2\right\}$$

$$= \frac{\sum_{i=1}^{n_A} e_{A_i}{}^2}{n_A} + \frac{\sum_{j=1}^{n_B} e_{B_j}{}^2}{n_B}$$

$$= \frac{SS_A}{n_A} + \frac{SS_B}{n_B}$$

$$\because \sum_{i=1}^{n_A} e_{A_i} = 0, \quad \sum_{j=1}^{n_B} e_{B_j} = 0, \quad \sum_{i=1}^{n_A}\sum_{j=1}^{n_B} e_{B_j}{}^2 = \sum_{j=1}^{n_B}\sum_{i=1}^{n_A} e_{B_j}{}^2$$

$$\sum_{j=1}^{n_B} e_{A_i}{}^2 = n_B e_{A_i}{}^2, \quad \sum_{i=1}^{n_A} e_{B_j}{}^2 = n_A e_{B_j}{}^2$$

$$\frac{\sum_{i=1}^{n_A}\sum_{j}^{n_B} e_{A_i}{}^2}{n_A n_B} = \frac{\sum_{i=1}^{n_A} e_{A_i}{}^2}{n_A} = \frac{SS_A}{n_A}$$

$$\frac{\sum_{i=1}^{n_A}\sum_{j}^{n_B} e_{B_j}{}^2}{n_A n_B} = \frac{\sum_{j=1}^{n_B} e_{B_j}{}^2}{n_B} = \frac{SS_B}{n_B}$$

Conclusively,

$$\frac{SS_{A+B}}{n_A n_B} = \frac{SS_A}{n_A} + \frac{SS_B}{n_B}$$

From this we can obtain formula 39

$$SS_{A+B} = n_B SS_{\hat{A}} + n_A SS_{\hat{B}}$$

More simply, when the structure of the data is as follow.

$$e_{A+B_{ij}} = e_{A_i} + e_{B_j}$$

Average is

$$\sum_{i=1}^{n_A}\sum_{j=1}^{n_B} e_{A+B_{ij}} = n_B \sum_{i=1}^{n_A} e_{A_i} + n_A \sum_{j=1}^{n_B} e_{B_j} = 0$$

Total SS is

$$\sum_{i=1}^{n_A}\sum_{j=1}^{n_B} e_{A+B_{ij}}{}^2 = n_B \sum_{i=1}^{n_A} e_{A_i}{}^2 + n_A \sum_{j=1}^{n_B} e_{B_j}{}^2$$

In the case, should consider interaction

Data structure is

$$e_{A+B+I_{ij}} = e_{A_i} + e_{B_j} + e_{I_{ij}}$$

Average is as follow

$$\sum_{i=1}^{n_A}\sum_{j=1}^{n_B} e_{A+B+I_{ij}} = n_B \sum_{i=1}^{n_A} e_{A_i} + n_A \sum_{j=1}^{n_B} e_{B_j} + \sum_{i=1}^{n_A}\sum_{j=1}^{n_B} e_{I_{ij}} = 0$$

$$I_{ij}\text{: interaction between } A_i \text{ and } B_j$$

SS is as follow.

$$\sum_{i=1}^{n_A}\sum_{j=1}^{n_B} e_{A+B+I_{ij}}{}^2 = \sum_{i=1}^{n_A}\sum_{j=1}^{n_B} \left(e_{A_i} + e_{B_j} + e_{I_{ij}}\right)^2$$

$$= \sum_{j=1}^{n_B}\sum_{i=1}^{n_A} e_{A+B_{ij}}{}^2 + 2\sum_{i=1}^{n_A}\sum_{j=1}^{n_B} e_{I_{ij}}\left(e_{A_i} + e_{B_j}\right) + \sum_{i=1}^{n_A}\sum_{j=1}^{n_B} e_{I_{ij}}{}^2$$

$$= n_B SS_A + n_A SS_B + 2\sum_{i=1}^{n_A} e_{A_i} \sum_{j=1}^{n_B} e_{I_{ij}} + 2\sum_{j=1}^{n_B} e_{B_j} \sum_{i=1}^{n_A} e_{I_{ij}} + + SS_I$$

$$= n_B SS_A + n_A SS_B + SS_I$$

$$\because \sum_{j=1}^{n_B} e_{I_{ij}} = \sum_{i=1}^{n_A} e_{I_{ij}} = 0$$

In the case we have to consider interaction and residual, the structure of the data is as follow.

$$e_{A+B+I+r_{ijk}} = e_{A_i} + e_{B_j} + e_{I_{ij}} + e_{r_{ijk}}$$

Average is as follow.

$$\sum_{i=1}^{n_A}\sum_{j=1}^{n_B}\sum_{k=1}^{n_r} e_{A+B+I+r_{ijk}} = n_r n_B \sum_{i=1}^{n_A} e_{A_i} + n_r n_A \sum_{j=1}^{n_B} e_{B_j} + n_r \sum_{i=1}^{n_A}\sum_{j=1}^{n_B} e_{I_{ij}} + \sum_{i=1}^{n_A}\sum_{j=1}^{n_B}\sum_{k=1}^{n_r} r_{ijk} = 0$$

Total SS is as follow

$$\sum_{i=1}^{n_A}\sum_{j=1}^{n_B}\sum_{k=1}^{n_r} e_{A+B+I+r_{ijk}}{}^2 = \sum_{i=1}^{n_A}\sum_{j=1}^{n_B}\sum_{k=1}^{n_r} \left(e_{A+B+I_{ijk}} + e_{r_{ijk}}\right)^2$$

$$= \sum_{k=1}^{n_r}\sum_{i=1}^{n_A}\sum_{j=1}^{n_B} {e_{A+B+I_{ijk}}}^2 + 2\sum_{i=1}^{n_A}\sum_{j=1}^{n_B}\sum_{k=1}^{n_r} e_{A+B+I_{ij}}\, e_{r_{ijk}} + \sum_{i=1}^{n_A}\sum_{j=1}^{n_B}\sum_{k=1}^{n_r} {e_{r_{ijk}}}^2$$

$$= n_r(n_B SS_A + n_A SS_B + SS_I) + 2\sum_{i=1}^{n_A}\sum_{j=1}^{n_B} e_{A+B+I_{ij}} \sum_{k=1}^{n_r} e_{r_{ijk}} + \sum_{i=1}^{n_A}\sum_{j=1}^{n_B}\sum_{k=1}^{n_r} {e_{r_{ijk}}}^2$$

$$= n_r(n_B SS_A + n_A SS_B + SS_I) + SS_r$$

$$\because \sum_{k=1}^{n_r} e_{r_{ijk}} = 0$$

Theoretically, we have reached the conclusion. However, it is better to consider the calculation procedure of SSs, because the structure is complicated and sometimes we are confused in the calculation process. There is a simplified procedure and it is better to use the simplified procedure to avoid unnecessary confusion.

The basic idea is to use the theory that total SS is sum of partial SSs

$$SS_{total} = n_r(n_B SS_A + n_A SS_B + SS_I) + SS_r$$

I: interaction A and B

and use the simplified calculation of SS

$$\mathrm{SS} = \mathrm{S} - \frac{T^2}{n}$$

1st step: calculation of T_{total}

		$i=1$	$i=2$	$\cdots$		$i=n_A$	Sum
$j=1$	$k=1$	d_{111}	d_{211}	d_{i11}		$d_{n_A 11}$	$\sum_{i=1}^{n_A} d_{i11}$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
	$k=n_r$	$d_{11\,n_r}$	$d_{21\,n_r}$	d_{i1n_r}		d_{n1n_r}	$\sum_{i=1}^{n_A} d_{i1n_r}$
$j=2$	$k=1$	d_{121}	d_{221}	d_{i21}		$d_{n_A 21}$	$\sum_{i=1}^{n_A} d_{i21}$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
	$k=n_r$	$d_{12\,n_r}$	$d_{22\,n_r}$	d_{i2n_r}		$d_{n_A 2n_r}$	$\sum_{i=1}^{n_A} d_{i2n_r}$

⋮	$k=1$ ⋮ $k=n_r$	d_{1j1} ⋮ $d_{1j\,n_r}$	d_{2j1} ⋮ $d_{2j\,n_r}$	d_{ij1} d_{ijk} d_{ijn_r}		d_{n_Aj1} ⋮ d_{njn_r}	$\sum_{i=1}^{n_A} d_{ij1}$ ⋮ $\sum_{i=1}^{n_A} d_{ijn_r}$
$j=n_B$	$k=1$ ⋮ $k=n_r$	d_{1n_B1} ⋮ $d_{1\,n_Bn_r}$	d_{2n_B1} ⋮ $d_{2\,n_Bn_r}$	d_{in_B1} ⋮ $d_{i\,n_Bn_r}$		$d_{n_An_B1}$ ⋮ $d_{n_An_Bn_r}$	$\sum_{i=1}^{n_A} d_{in_B1}$ ⋮ $\sum_{i=1}^{n_A} d_{in_Bn_r}$
Sum		$T_{total} = \sum_{i=1}^{n_A}\sum_{j=1}^{n_B}\sum_{k=1}^{n_r} d_{ijk}$					

Example

	A_1	A_2	A_3	sum
B_1	3.8	6.9	7.3	18
	3.4	6.8	7.8	18
	3	7.3	7.7	18
B_2	7.3	10.8	11.9	30
	6.8	10.6	12.6	30
	6.6	11	12.4	30
B_3	7.7	12.2	13.2	33.1
	7.6	12.2	12.9	32.7

	7.8	12.2	13.2	33.2
	10.1	13.9	14.9	38.9
B_4	9.9	13.9	14.7	38.5
	10	14.2	15.4	39.6
Sum	84	132	144	360

$$\mathrm{T}_{total} = 360$$

2nd step: Calculation of S_{total}

		$i = 1$	$i = 2$	$\cdots$	$i = n_A$	Sum
	$k = 1$	${d_{111}}^2$	${d_{211}}^2$	${d_{i1}}^2$	${d_{n_A 11}}^2$	$\sum_{i=1}^{n_A} {d_{i11}}^2$
$j = 1$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$k = n_r$	${d_{11\,n_r}}^2$	${d_{21\,n_r}}^2$	${d_{i1n_r}}^2$	${d_{n1n_r}}^2$	$\sum_{i=1}^{n_A} {d_{i1n_r}}^2$
	$k = 1$	${d_{121}}^2$	${d_{221}}^2$	${d_{i2}}^2$	${d_{n_A 21}}^2$	$\sum_{i=1}^{n_A} {d_{i21}}^2$
$j = 2$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$k = n_r$	${d_{12\,n_r}}^2$	${d_{22\,n_r}}^2$	${d_{i2n_r}}^2$	${d_{n2n_r}}^2$	$\sum_{i=1}^{n_A} {d_{i2n_r}}^2$
	$k = 1$	${d_{1j1}}^2$	${d_{2j1}}^2$	${d_{ij1}}^2$	${d_{n_A j1}}^2$	$\sum_{i=1}^{n_A} {d_{ij1}}^2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	${d_{ijk}}^2$	$\vdots$	$\vdots$
	$k = n_r$	${d_{1j\,n_r}}^2$	${d_{2j\,n_r}}^2$	${d_{ijn_r}}^2$	${d_{nj\quad r}}^2$	$\sum_{i=1}^{n_A} {d_{ij\quad r}}^2$

$j = n_B$	$k = 1$	${d_{1n_B1}}^2$	${d_{2n_B1}}^2$	${d_{in_B1}}^2$	${d_{n_An_B1}}^2$	$\sum_{i=1}^{n_A} {d_{in_B1}}^2$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$k = n_r$	${d_{1\,n_Bn_r}}^2$	${d_{2\,n_Bn_r}}^2$	${d_{i\,n_Bn_r}}^2$	${d_{n_An_Bn_r}}^2$	$\sum_{i=1}^{n_A} {d_{in_Bn_r}}^2$
Sum	$S_{total} = \sum_{i=1}^{n_A}\sum_{j=1}^{n_B}\sum_{k=1}^{n_r} {d_{ijk}}^2$					

Example

	A1	A2	A3	Sum
B_1	14.44	47.61	53.29	115.34
	11.56	46.24	60.84	118.64
	9	53.29	59.29	121.58
B_2	53.29	116.64	141.61	311.54
	46.24	112.36	158.76	317.36
	43.56	121	153.76	318.32
B_3	59.29	148.84	174.24	382.37
	57.76	148.84	166.41	373.01
	60.84	148.84	174.24	383.92
B_4	102.01	193.21	222.01	517.23
	98.01	193.21	216.09	507.31
	100	201.64	237.16	538.8
Sum	656	1531.72	1817.7	4005.42

$$S_{total} = 4005.42$$

$$\mathrm{SS}_{total} = \mathrm{S}_{total} - \frac{{T_{total}}^2}{n_A n_B n_r} = 4005.42 - \frac{360^2}{36} = 405.42$$

	$i=1$	$i=2$	$\cdots$	$i=n_A$	Sum
$j=1$	T_{11} S_{11} $\mathrm{S}_{11}-\frac{{T_{11}}^2}{n_r}$	T_{21} S_{21} $\mathrm{S}_{21}-\frac{{T_{21}}^2}{n_r}$	T_{i1} S_{i1} $\mathrm{S}_{i1}-\frac{{T_{i1}}^2}{n_r}$	T_{n_A1} S_{n_A1} $\mathrm{S}_{n_A1}-\frac{{T_{n_A1}}^2}{n_r}$	$\sum_{i=1}^{n_A}\left(\mathrm{S}_{i1}-\frac{{T_{i1}}^2}{n_r}\right)$
$j=2$	T_{12} S_{12} $\mathrm{S}_{12}-\frac{{T_{12}}^2}{n_r}$	T_{22} S_{22} $\mathrm{S}_{22}-\frac{{T_{22}}^2}{n_r}$	T_{i2} S_{i2} $\mathrm{S}_{i2}-\frac{{T_{i2}}^2}{n_r}$	T_{n_A2} S_{n_A2} $\mathrm{S}_{n_A2}-\frac{{T_{n_A2}}^2}{n_r}$	$\sum_{i=1}^{n_A}\left(\mathrm{S}_{i2}-\frac{{T_{i2}}^2}{n_r}\right)$
$\vdots$	T_{1j} S_{1j} $\mathrm{S}_{1j}-\frac{{T_{1j}}^2}{n_r}$	T_{2j} S_{2j} $\mathrm{S}_{2j}-\frac{{T_{22}}^2}{n_r}$	T_{ij} S_{ij} $\mathrm{S}_{ij}-\frac{{T_{ij}}^2}{n_r}$	T_{n_Aj} S_{n_Aj} $\mathrm{S}_{n_Aj}-\frac{{T_{n_Aj}}^2}{n_r}$	$\sum_{i=1}^{n_A}\left(\mathrm{S}_{ij}-\frac{{T_{ij}}^2}{n_r}\right)$
$j=n_B$	T_{1n_B} S_{1n_B} $\mathrm{S}_{1n_B}-\frac{{T_{1n_B}}^2}{n_r}$	T_{2n_B} S_{2n_B} $\mathrm{S}_{2j}-\frac{{T_{2n_B}}^2}{n_r}$	T_{in_B} S_{in_B} $\mathrm{S}_{in_B}-\frac{{T_{in_B}}^2}{n_r}$	$T_{n_An_B}$ $\mathrm{S}_{n_An_B}$ $\mathrm{S}_{n_An_B}-\frac{{T_{n_An_B}}^2}{n_r}$	$\sum_{i=1}^{n_A}\left(\mathrm{S}_{in_B}-\frac{{T_{in_B}}^2}{n_r}\right)$
Sum	$\sum_{I=1}^{n_B}\sum_{I=1}^{n_A}\sum_{k=1}^{n_r}S_{ijr}-\frac{1}{n_r}\sum_{j=1}^{n_B}\sum_{i=1}^{n_A}\left(\sum_{k=1}^{n_r}d_{ihk}\right)^2$				

$$\mathrm{SS}_{residual}=\sum_{I=1}^{n_B}\sum_{I=1}^{n_A}\sum_{k=1}^{n_r}S_{ijr}-\frac{1}{n_r}\sum_{j=1}^{n_B}\sum_{i=1}^{n_A}\left(\sum_{k=1}^{n_r}d_{ihk}\right)^2=S_{total}-\frac{1}{n_r}\sum_{j=1}^{n_B}\sum_{i=1}^{n_A}\left(\sum_{k=1}^{n_r}d_{ihk}\right)^2$$

$$=S_{total}-\frac{1}{n_r}\sum_{j=1}^{n_B}\sum_{i=1}^{n_A}{T_{ij}}^2$$

Here,

$$T_{ij}=\sum_{k=1}^{n_r}d_{ihk}$$

Using this, we can calculate $\sum_{j=1}^{n_B}\sum_{i=1}^{n_A}{T_{ij}}^2$ more simplified procedure

	$i=1$	$i=2$	$\cdots$	$i=n_A$	Sum
$j=1$	T_{11}	T_{21}	T_{i1}	T_{n_A1}	$\sum_{i=1}^{n_A} T_{i1}{}^2$
	$T_{11}{}^2$	$T_{21}{}^2$	$T_{i1}{}^2$	$T_{n_A1}{}^2$	
$j=2$	T_{12}	T_{22}	T_{i2}	T_{n_A2}	$\sum_{i=1}^{n_A} T_{i2}{}^2$
	$T_{12}{}^2$	$T_{22}{}^2$	$T_{i2}{}^2$	$T_{n_A2}{}^2$	
$\vdots$	T_{1j}	T_{2j}	T_{ij}	T_{n_Aj}	$\sum_{i=1}^{n_A} T_{ij}{}^2$
	$T_{1j}{}^2$	$T_{22}{}^2$	$T_{ij}{}^2$	$T_{n_Aj}{}^2$	
$j=n_B$	T_{1n_B}	T_{2n_B}	T_{in_B}	$T_{n_An_B}$	$\sum_{i=1}^{n_A} T_{in_B}{}^2$
	$T_{1n_B}{}^2$	$T_{2n_B}{}^2$	$T_{in_B}{}^2$	$T_{n_An_B}{}^2$	
Sum	$\sum_{j=1}^{n_B}\sum_{i=1}^{n_A} T_{ij}{}^2$				

Example

	A1	A2	A3	sum
B1	3.8	6.9	7.3	
	3.4	6.8	7.8	
	3	7.3	7.7	
sum	10.2	21	22.8	
T^2	104.04	441	519.84	1064.88
B2	7.3	10.8	11.9	
	6.8	10.6	12.6	
	6.6	11	12.4	
sum	20.7	32.4	36.9	
T^2	428.49	1049.76	1361.61	2839.86
B3	7.7	12.2	13.2	
	7.6	12.2	12.9	
	7.8	12.2	13.2	
sum	23.1	36.6	39.3	
T^2	533.61	1339.56	1544.49	3417.66
B4	10.1	13.9	14.9	
	9.9	13.9	14.7	
	10	14.2	15.4	
sum	30	42	45	
T^2	900	1764	2025	4689
Sum				12011.4

$$SS_{residual} = S_{total} - \frac{1}{n_r}\sum_{j=1}^{n_B}\sum_{i=1}^{n_A} {T_{ij}}^2$$

$$SS_{residual} = 4005.42 - \frac{12011.4}{3} = 4005.42 - 4003.8 = 1.62$$

4th step: Calculation of SS_B

$$n_r n_A SS_B = SS_{total} - n_r(n_B SS_A + SS_I) + SS_r$$

The meaning of $n_r(n_B SS_A + SS_I) + SS_r$is SS without SS_B.

When we express the SS as $SS_{total-B}$, $SS_{total-B}$ is obtainable by summing up the SS in each condition of factor B.

The Calculation is as follow

		$i = 1$	$i = 2$	$\cdots$	$i = n_A$	Sum
$j = 1$	T	T_{11}	T_{21}	T_{i1}	$T_{n_A 1}$	$\left(\sum_{i=1}^{n_A} T_{i1}\right)^2$
	S	S_{11}	S_{21}	S_{i1}	$S_{n_A 1}$	$\sum_{i=1}^{n_A}\sum_{k=1}^{n_r} {d_{i1k}}^2$
	SS	$\sum_{i=1}^{n_A}\sum_{k=1}^{n_r} {d_{i1k}}^2 - \frac{1}{n_A n_r}\left(\sum_{i=1}^{n_A} T_{i1}\right)^2$				
$j = 2$	T	T_{12}	T_{22}	T_{i2}	$T_{n_A 2}$	$\left(\sum_{i=1}^{n_A} T_{i2}\right)^2$
	S	S_{12}	S_{22}	S_{i2}	$S_{n_A 2}$	$\sum_{i=1}^{n_A}\sum_{k=1}^{n_r} {d_{i2k}}^2$
	SS	$\sum_{i=1}^{n_A}\sum_{k=1}^{n_r} {d_{i2k}}^2 - \frac{1}{n_A n_r}\left(\sum_{i=1}^{n_A} T_{i2}\right)^2$				
$\vdots$	T	T_{1j}	T_{2j}	T_{ij}	$T_{n_A j}$	$\left(\sum_{i=1}^{n_A} T_{ij}\right)^2$
	S	S_{1j}	S_{2j}	S_{ij}	S_{j2}	$\sum_{i=1}^{n_A}\sum_{k=1}^{n_r} {d_{ijk}}^2$
	SS	$\sum_{i=1}^{n_A}\sum_{k=1}^{n_r} {d_{ijk}}^2 - \frac{1}{n_A n_r}\left(\sum_{i=1}^{n_A} T_{ij}\right)^2$				
$j = n_B$	T	T_{1n_B}	T_{2n_B}	T_{in_B}	$T_{n_A n_B}$	$\left(\sum_{i=1}^{n_A} T_{in_B}\right)^2$
	S	S_{1j}	S_{2j}	S_{ij}	S_{j2}	$\sum_{i=1}^{n_A}\sum_{k=1}^{n_r} {d_{ijk}}^2$
	SS	$\sum_{i=1}^{n_A}\sum_{k=1}^{n_r} {d_{ijk}}^2 - \frac{1}{n_A n_r}\left(\sum_{i=1}^{n_A} T_{ij}\right)^2$				
$SS_{total-B}$		$\sum_{jz-1}^{n_B}\sum_{i=1}^{n_A}\sum_{k=1}^{n_r} {d_{ijk}}^2 - \frac{1}{n_A n_r}\sum_{j=1}^{n_B}\left(\sum_{i=1}^{n_A} T_{ij}\right)^2$				

$$\sum_{jz-1}^{n_B}\sum_{i=1}^{n_A}\sum_{k=1}^{n_r} {d_{ijk}}^2 = S_{total}$$

$$SS_{total} \quad = \sum_{jz-1}^{n_B}\sum_{i=1}^{n_A}\sum_{k=1}^{n_r} {d_{ijk}}^2 - \frac{1}{n_A n_r}\sum_{j=1}^{n_B}\left(\sum_{i=1}^{n_A} T_{ij}\right)^2 = S_{total} - \frac{1}{n_A n_r}\sum_{j=1}^{n_B}\left(\sum_{i=1}^{n_A} T_{ij}\right)^2$$

$$n_r n_A SS_B = SS_{total} - SS_{total-B} = S_{total} - \frac{{T_{total}}^2}{n_A n_b n_r} - \left(S_{total} - \frac{1}{n_A n_r}\sum_{j=1}^{n_B}\left(\sum_{i=1}^{n_A} T_{ij}\right)^2\right)$$

$$= \frac{1}{n_A n_r}\sum_{j=1}^{n_B}\left(\sum_{i=1}^{n_A} T_{ij}\right)^2 - \frac{1}{n_A n_b n_r}\left(\sum_{i=1}^{n_A}\sum_{j=1}^{n_B}\sum_{k=1}^{n_r} d_{ijk}\right)^2$$

Calculation of $\sum_{j=1}^{n_B}\left(\sum_{i=1}^{n_A} T_{ij}\right)^2$

	$i = 1$	$i = 2$	$\cdots$	$i = n_A$	Sum
$j = 1$	T_{11}	T_{21}	T_{i1}	$T_{n_A 1}$	$\left(\sum_{i=1}^{n_A} T_{i1}\right)^2$
$j = 2$	T_{12}	T_{22}	T_{i2}	$T_{n_A 2}$	$\left(\sum_{i=1}^{n_A} T_{i2}\right)^2$
$\vdots$	T_{1j}	T_{2j}	T_{ij}	$T_{n_A j}$	$\left(\sum_{i=1}^{n_A} T_{ij}\right)^2$
$j = n_B$	T_{1n_B}	T_{2n_B}	T_{in_B}	$T_{n_A n_B}$	$\left(\sum_{i=1}^{n_A} T_{in_B}\right)^2$
Sum	$\sum_{j=1}^{n_B}\left(\sum_{i=1}^{n_A} T_{ij}\right)^2$				

Example

	A1	A2	A3	sum	T_j^2
B1	3.8	6.9	7.3	54	2916
	3.4	6.8	7.8		
	3	7.3	7.7		
B2	7.3	10.8	11.9	90	8100
	6.8	10.6	12.6		
	6.6	11	12.4		
B3	7.7	12.2	13.2	99	9801
	7.6	12.2	12.9		
	7.8	12.2	13.2		
B4	10.1	13.9	14.9	117	13689
	9.9	13.9	14.7		
	10	14.2	15.4		
Sum					34506

$$n_r n_A SS_B = \frac{1}{n_A n_r}\sum_{j=1}^{n_B}\left(\sum_{i=1}^{n_A} T_{ij}\right)^2 - \frac{1}{n_A n_b n_r}\left(\sum_{i=1}^{n_A}\sum_{j=1}^{n_B}\sum_{k=1}^{n_r} d_{ijk}\right)^2$$

$$= \frac{34506}{9} - \frac{360^2}{36} = 3834 - 3600 = 234$$

$$3 \times 3 \times SS_B = 234$$

$$SS_B = 26$$

5th step: Calculation of SS_A

Similar as

$$n_r n_B SS_A = SS_{total} - n_r(n_A SS_B + SS_I) + SS_r$$

The meaning of $n_r(n_A SS_B + SS_I) + SS_r$ is SS without SS_A.

When we express the SS as SS_{total} , $SS_{total-A}$ is obtainable by summing up the SS in each condition of factor A.

$$SS_{total-A} = \sum_{jz-1}^{n_B}\sum_{i=1}^{n_A}\sum_{k=1}^{n_r} {d_{ijk}}^2 - \frac{1}{n_B n_r}\sum_{i=1}^{n_A}\left(\sum_{j=1}^{n_B} T_{ij}\right)^2 = S_{total} - \frac{1}{n_B n_r}\sum_{j=1}^{n_A}\left(\sum_{j=1}^{n_B} T_{ij}\right)^2$$

$$n_r n_B SS_A = SS_{total} - SS_{total-A} = S_{total} - \frac{{T_{total}}^2}{n_A n_B n_r} - \left(S_{total} - \frac{1}{n_B n_r} \sum_{j=1}^{n_A} \left(\sum_{i=1}^{n_B} T_{ij} \right)^2 \right)$$

$$= \frac{1}{n_B n_r} \sum_{j=1}^{n_a} \left(\sum_{i=1}^{n_B} T_{ij} \right)^2 - \frac{1}{n_A n_B n_r} \left(\sum_{i=1}^{n_A} \sum_{j=1}^{n_B} \sum_{k=1}^{n_r} d_{ijk} \right)^2$$

Calculation of $\sum_{j=1}^{n_a} \left(\sum_{i=1}^{n_B} T_{ij} \right)^2$

Example

	A1	A2	A3	Sum
B1	3.8	6.9	7.3	
	3.4	6.8	7.8	
	3	7.3	7.7	
B2	7.3	10.8	11.9	
	6.8	10.6	12.6	
	6.6	11	12.4	
B3	7.7	12.2	13.2	
	7.6	12.2	12.9	
	7.8	12.2	13.2	
B4	10.1	13.9	14.9	
	9.9	13.9	14.7	
	10	14.2	15.4	
Sum	84	132	144	
	7056	17424	20736	45216

$$n_r n_B SS_A = \frac{1}{n_B n_r} \sum_{j=1}^{n_a} \left(\sum_{i=1}^{n_B} T_{ij} \right)^2 - \frac{1}{n_A n_b n_r} \left(\sum_{i=1}^{n_A} \sum_{j=1}^{n_B} \sum_{k=1}^{n_r} d_{ijk} \right)^2$$

$$= \frac{45216}{12} - \frac{360^2}{36} = 3768 - 3600 = 168$$

$$3 \times 4 \times SS_A = 168$$

$$SS_A = 14$$

6th step: Calculation of $\mathrm{SS}_{interaction(A \times B)}$

$$SS_{total} = n_r(n_B SS_A + n_A SS_B + SS_I) + SS_r$$

$$n_r SS_I = SS_{total} - (nr \quad (n_B SS_A + n_A SS_B + SS_I) + SS_r)$$

$$= 405.42 - 168 - 234 - 1.62 = 1.8$$

$$3 \times SS_I = 1.8$$

$$SS_I = 0.6$$

Degree of freedom

$$\mathrm{df}_{total} = n_A n_B n_r - 1 = 35$$

$$\mathrm{df}_A = n_A - 1 = 2$$

$$\mathrm{df}_b = n_B - 1 = 3$$

$$\mathrm{df}_{interaction(A\cdot B)} = (n_A - 1)(n_B - 1) = 2 \times 3 = 6$$

$$\mathrm{df}_{residual} = (n_r - 1)n_A n_B = 2 \times 3 \times 4 = 24$$

$$\mathrm{df}_{total} = \mathrm{df}_A + \mathrm{df}_b + \mathrm{df}_{interaction(A\cdot B)} + \mathrm{df}_{residual}$$

Summery

$$\mathrm{SS}_{total} = \sum_{jz-1}^{n_B}\sum_{i=1}^{n_A}\sum_{k=1}^{n_r} {d_{ijk}}^2 - \frac{1}{n_A n_b n_r}\left(\sum_{i=1}^{n_A}\sum_{j=1}^{n_B}\sum_{k=1}^{n_r} d_{ijk}\right)^2 = \mathrm{S}_{total} - \frac{{T_{total}}^2}{n_A n_B n_r}$$

$$\mathrm{S}_{total} = \sum_{jz-1}^{n_B}\sum_{i=1}^{n_A}\sum_{k=1}^{n_r} {d_{ijk}}^2, \qquad {T_{total}}^2 = \left(\sum_{i=1}^{n_A}\sum_{j=1}^{n_B}\sum_{k=1}^{n_r} d_{ijk}\right)^2$$

$$\mathrm{SS}_{resudual} = \sum_{jz-1}^{n_B}\sum_{i=1}^{n_A}\sum_{k=1}^{n_r} {d_{ijk}}^2 - \frac{1}{n_r}\sum_{j=1}^{n_B}\sum_{i=1}^{n_A}\left(\sum_{k=1}^{n_r} d_{ijk}\right)^2 = \mathrm{S}_{total} - \frac{{T_{residual}}^2}{n_A n_B n_r}$$

$$n_B n_r \mathrm{SS}_A = \frac{1}{n_B n_r}\sum_{j=1}^{n_A}\left(\sum_{i=1}^{n_B} T_{ij}\right)^2 - \frac{{T_{total}}^2}{n_A n_B n_r} = \frac{T_{Total-A}}{n_B n_r} - \frac{{T_{total}}^2}{n_A n_B n_r}$$

$$\because T_{Tot} \qquad = \sum_{j=1}^{n_A}\left(\sum_{i=1}^{n_B} T_{ij}\right)^2$$

$$n_A n_r \mathrm{SS}_B = \frac{1}{n_A n_r}\sum_{j=1}^{n_B}\left(\sum_{i=1}^{n_A} T_{ij}\right)^2 - \frac{{T_{total}}^2}{n_A n_B n_r} = \frac{T_{Total-B}}{n_A n_r} - \frac{{T_{total}}^2}{n_A n_B n_r}$$

$$\because \frac{T_{Total-A}}{n_A n_r} = \sum_{j=1}^{n_B}\left(\sum_{i=1}^{n_A} T_{ij}\right)^2$$