V-1. Fundamental matrix calculation

V-1-1. What is matrix

Mathematics of matrix has been developed in various field for various purposes. It has various interpretation and application situation. It is important to know the meaning and function of matrix in deeper level. However, the author is thinking that readers who have aversion to mathematics get tired of learning matrix hearing such foggy and horror word. Most comfortable entry to world of matrix is to consider that matrix is arithmetic system to solve simultaneous equation. Mathematicians may criticize the opinion of author as too easy explanation. Learning at ease is policy of this text book. Let us remain complicate and difficult things for future learning. Getting experiences everyone will consider complicate and difficult issues after use of the tools.

We consider following simultaneous equation

$$ax + by = \alpha$$

$$kx + ly = \beta$$

We can express the equation using matrix as follow.

$$\begin{pmatrix} ax + by \\ kx + ly \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

The left side can be express as product of two matrix as follow.

$$\begin{pmatrix} a & b \\ k & l \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

A set of numbers enclosed by a parenthesis is a matrix. Matrix is a set of numbers composed from row and column. Some readers may not understand why we should do so. Please accept this as easy express method of simultaneous equation.
Using this notation method

$$ax + by + cz = \alpha$$

$$kx + ly + mz = \beta$$

$$sx + ty + uz = \gamma$$

can be expressed as follow.

$$\begin{pmatrix} a & b & c \\ k & l & m \\ s & t & u \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

The author thinks this is convenient.
Let us solve the simultaneous equation.

$$ax + by = \alpha$$

$$kx + ly = \beta$$

Step by step calculation procedure is as follow.

$$akx + bky = k\alpha$$

$$akx + aly = a\beta$$

$$(bk - al)y = k\alpha - a\beta$$

$$y = \frac{k\alpha - a\beta}{-(al - bk)}$$

Put this in

$$ax + by = \alpha$$

$$ax + b\frac{k\alpha - a\beta}{-(al - bk)} = \alpha$$

$$ax = \alpha + b\frac{k\alpha - a\beta}{al - bk}$$

$$ax = \frac{al\alpha - bk\alpha + bk\alpha - ab\beta}{al - bk}$$

$$ax = \frac{al\alpha - ab\beta}{al - bk}$$

$$x = \frac{l\alpha - b\beta}{al - bk}$$

Answer

$$x = \frac{l\alpha - b\beta}{al - bk}$$

$$y = \frac{-(k\alpha - a\beta)}{al - bk}$$

Dominator of right side is determinant of matrix $\begin{pmatrix} a & b \\ k & l \end{pmatrix}$

The author is understanding that readers are confusing pushed such information suddenly without prior knowledge. Many readers do not know what is determinant. The author suggest reader to accept determinant as size (magnitude) of matrix. Determinant of matrix is expressed as follow

$$\begin{vmatrix} a & b \\ k & l \end{vmatrix}$$

The author will explain meaning of the magnitude of matrix in later paragraphs.
Before that calculation of determinant is as follow.

$$\begin{vmatrix} a & b \\ k & l \end{vmatrix} = al - bk$$

The solution of the simultaneous equation expressed using determinant is as follow

$$x = \frac{l\alpha - b\beta}{\begin{vmatrix} a & b \\ k & l \end{vmatrix}}$$

$$y = \frac{-(k\alpha - a\beta)}{\begin{vmatrix} a & b \\ k & l \end{vmatrix}}$$

This is an example of two unknowns. Example of three unknowns is as follow.

$$\begin{pmatrix} a & b & c \\ k & l & m \\ s & t & u \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$\begin{vmatrix} a & b & c \\ k & l & m \\ s & t & u \end{vmatrix} = alu + bms + ckt - amt - bku - cls$$

This is determinant of 3×3 matrix. Most of readers want to know calculation of determinant of $n \times n$ matrix. That is only procedure of calculation. Most important thing is sensuous understanding fundamental idea of calculation of matrix. The author introduces concept of determinant at first, and then the calculation procedure　I later paragraphs.

In case of one unknown,
Expression of the relation is

$$ax = \alpha$$

Then the solution is

$$x = \frac{\alpha}{a} = a^{-1}\alpha$$

We want to express the relation of simultaneous equation of n unknown as

$$\boldsymbol{AX} = \boldsymbol{\alpha}$$

In the case of 3 unknowns,

$$\boldsymbol{A} = \begin{pmatrix} a & b & c \\ k & l & m \\ s & t & u \end{pmatrix}$$

$$\boldsymbol{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$\boldsymbol{AX} = \boldsymbol{\alpha}$$

We can express complicate simultaneous equation in one line by this notation system,
We also want to simplify the procedure of calculation.

In the case of one unknown,

$$ax = \alpha$$

$$x = \frac{\alpha}{a}$$

We divide right side by coefficient of left side. This means multiplying inverse of coefficient to both sides.

$$ax = \alpha$$

$$a^{-1}ax = a^{-1}\alpha$$

$$1 \cdot x = a^{-1}\alpha$$

$$x = \frac{\alpha}{a}$$

In the case of matrix calculation, if there exist $\boldsymbol{A^{-1}}$, we can solve following simultaneous equation

$$\boldsymbol{A} = \begin{pmatrix} a & b & c \\ k & l & m \\ s & t & u \end{pmatrix}$$

$$\boldsymbol{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$\boldsymbol{AX} = \boldsymbol{\alpha}$$

By multiplying $\boldsymbol{A^{-1}}$ to both sides as follow.

$$\boldsymbol{A^{-1}AX} = \boldsymbol{A^{-1}\alpha}$$

$$\boldsymbol{IX} = \boldsymbol{A^{-1}\alpha}$$

$$\boldsymbol{X} = \boldsymbol{A^{-1}\alpha}$$

I is something like1 in case of one unknown, which makes no change.

Here what we should do are

1) Consider calculation procedure of matrix
2) Consider inverse of matrix
3) Consider unit of matrix

The name of inverse of matrix is inverse matrix, and the name of unit of matrix is identity matrix.

We consider from simplest case as usual.

In the case of following two unknown simultaneous equation.

$$ax + by = \alpha$$

$$kx + ly = \beta$$

Followings are solution

$$x = \frac{l\alpha - b\beta}{al - bk}$$

$$y = \frac{-(k\alpha - a\beta)}{al - bk}$$

In the form of matrix calculation

$$\begin{pmatrix} a & b \\ k & l \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ k & l \end{pmatrix}$$

$$\boldsymbol{X} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Multiply$\boldsymbol{A}^{-1}$ to both side

$$\boldsymbol{A}^{-1}\boldsymbol{A}\boldsymbol{X} = \boldsymbol{A}^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

We do not know what is $\boldsymbol{A}^{-1}\boldsymbol{A}$**.** However, the function of $\boldsymbol{A}^{-1}\boldsymbol{A}$ is making no change. So,

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$$

$$\boldsymbol{X} = \boldsymbol{A}^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\boldsymbol{X} = \begin{pmatrix} x \\ y \end{pmatrix} = \boldsymbol{A}^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \dfrac{l\alpha - b\beta}{al - bk} \\ \dfrac{-(k\alpha - a\beta)}{al - bk} \end{pmatrix}$$

Dominator of right side is determinant. Push out the determinant to outside of parenthesis.

$$\boldsymbol{X} = \frac{1}{\begin{bmatrix} a & b \\ k & l \end{bmatrix}}\begin{pmatrix} l\alpha - b\beta \\ -(k\alpha - a\beta) \end{pmatrix}$$

When we remove α and β from the parenthesis.

$$\begin{pmatrix} l & -b \\ -k & a \end{pmatrix}$$

Applying the calculation rule of matrix, we can obtain following equation.

$$\begin{pmatrix} l & -b \\ -k & a \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} l\alpha - b\beta \\ -k\alpha + a\beta \end{pmatrix}$$

And inverse matrix is

$$A^{-1} = \frac{\begin{pmatrix} l & -b \\ -k & a \end{pmatrix}}{\begin{bmatrix} a & b \\ k & l \end{bmatrix}}$$

Using these rules we solve

$$AX = \binom{\alpha}{\beta}$$

As follow

$$A^{-1}AX = A^{-1}\binom{\alpha}{\beta}$$

$$I\,X = A^{-1}\binom{\alpha}{\beta}$$

$$X = A^{-1}\binom{\alpha}{\beta}$$