V-1-10. Separation of matrix

Sometimes, we need to separate matrix to partial matrixes. Here, we consider how the inverse matrix can be expressed by original partial matrixes.

$$\boldsymbol{A}=\begin{pmatrix}\boldsymbol{A}_{11} & \boldsymbol{A}_{12}\\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22}\end{pmatrix}$$

$$\boldsymbol{A}=\left(\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1q} & a_{1q+1} & \cdots & a_{1n}\\ \vdots & \boldsymbol{A}_{11} & \vdots & \vdots & \boldsymbol{A}_{12} & \vdots\\ a_{p1} & \cdots & a_{pq} & a_{pq+1} & \cdots & a_{pn}\\ \hline a_{p+11} & \cdots & a_{p+11} & a_{p+1\,q+1} & \cdots & a_{p+1\,q+1}\\ \vdots & \boldsymbol{A}_{21} & \vdots & \vdots & \boldsymbol{A}_{22} & \vdots\\ a_{n1} & \cdots & a_{nq} & a_{nq+1} & \cdots & a_{nn}\end{array}\right)$$

$$\boldsymbol{B}=\begin{pmatrix}\boldsymbol{B}_{11} & \boldsymbol{B}_{12}\\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22}\end{pmatrix}$$

$$\boldsymbol{B}=\left(\begin{array}{ccc|ccc} b_{11} & \cdots & b_{1p} & b_{1p+1} & \cdots & b_{1n}\\ \vdots & \boldsymbol{B}_{11} & \vdots & \vdots & \boldsymbol{B}_{12} & \vdots\\ b_{q1} & \cdots & b_{qp} & b_{qp+1} & \cdots & b_{qn}\\ \hline b_{q+1\,1} & \cdots & b_{q+1\,p} & b_{q+1p+1} & \cdots & b_{q+1n}\\ \vdots & \boldsymbol{B}_{21} & \vdots & \vdots & \boldsymbol{B}_{22} & \vdots\\ b_{n1} & \cdots & b_{n\,p} & b_{n\,p+1} & \cdots & b_{n\,n}\end{array}\right)$$

$$\boldsymbol{AB}=\begin{pmatrix}\boldsymbol{A}_{11} & \boldsymbol{A}_{12}\\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22}\end{pmatrix}\begin{pmatrix}\boldsymbol{B}_{11} & \boldsymbol{B}_{12}\\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22}\end{pmatrix}=\begin{pmatrix}\boldsymbol{A}_{11}\boldsymbol{B}_{11}+\boldsymbol{A}_{12}\boldsymbol{B}_{21} & \boldsymbol{A}_{11}\boldsymbol{B}_{12}+\boldsymbol{A}_{12}\boldsymbol{B}_{22}\\ \boldsymbol{A}_{21}\boldsymbol{B}_{11}+\boldsymbol{A}_{22}\boldsymbol{B}_{21} & \boldsymbol{A}_{21}\boldsymbol{B}_{12}+\boldsymbol{A}_{22}\boldsymbol{B}_{22}\end{pmatrix}$$

$$\boldsymbol{A}_{11}\boldsymbol{B}_{11}+\boldsymbol{A}_{12}\boldsymbol{B}_{21}: p\times p \text{ square matrix}$$

$$\boldsymbol{A}_{21}\boldsymbol{B}_{12}+\boldsymbol{A}_{22}\boldsymbol{B}_{22}: (n-p)\times\ (n-p) \text{ square matrix}$$

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Example

$$\begin{pmatrix}1 & 2 & 4 & 5 & 1\\ 0 & 2 & 1 & 1 & 3\\ 3 & 0 & 2 & 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 2\\ 0 & 3\\ 1 & -2\\ -1 & 1\\ 2 & -3\end{pmatrix}$$

Weseparate the matrixes as follow. The number of rows of first matrix and the number of columun of second matrix is the same. Thus, we can multiply first and scond matirixes. For the caludulation, we make the separateion of rows in second matirixe sme as separation of column in first amtrix as follow.

$$\left(\begin{array}{cc|c|cc}1 & 2 & 4 & 5 & 1\\ \hline 0 & 2 & 1 & 1 & 3\\ 3 & 0 & 2 & 0 & 1\end{array}\right)\left(\begin{array}{cc}-1 & 2\\ 0 & 3\\ \hline 1 & -2\\ \hline -1 & 1\\ 2 & -3\end{array}\right)。$$

Caculation of partial matrixes is as follow.

$$\begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \boldsymbol{A}_{13} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \boldsymbol{A}_{23} \end{pmatrix} \begin{pmatrix} \boldsymbol{B}_{11} \\ \boldsymbol{B}_{21} \\ \boldsymbol{B}_{31} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_{11}\boldsymbol{B}_{11} + \boldsymbol{A}_{12}\boldsymbol{B}_{21} + \boldsymbol{A}_{13}\boldsymbol{B}_{31} \\ \boldsymbol{A}_{21}\boldsymbol{B}_{11} + \boldsymbol{A}_{22}\boldsymbol{B}_{21} + \boldsymbol{A}_{23}\boldsymbol{B}_{31} \end{pmatrix}$$

$$\boldsymbol{A}_{11} = \begin{pmatrix} 1 & 2 \end{pmatrix}$$

$$\boldsymbol{A}_{12} = \begin{pmatrix} 4 \end{pmatrix}$$

$$\boldsymbol{A}_{13} = \begin{pmatrix} 5 & 1 \end{pmatrix}$$

$$\boldsymbol{A}_{21} = \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$$

$$\boldsymbol{A}_{22} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\boldsymbol{A}_{23} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

$$\boldsymbol{B}_{11} = \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}$$

$$\boldsymbol{B}_{21} = \begin{pmatrix} 1 & -2 \end{pmatrix}$$

$$\boldsymbol{B}_{31} = \begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix}$$

We implement multiplication of partial matrixes.

$$\boldsymbol{A}_{11}\boldsymbol{B}_{11} + \boldsymbol{A}_{12}\boldsymbol{B}_{21} + \boldsymbol{A}_{13}\boldsymbol{B}_{31} = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 4 \end{pmatrix} \begin{pmatrix} 1 & -2 \end{pmatrix} + \begin{pmatrix} 5 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \times (-1) + 2 \times 0 & 1 \times 2 + 2 \times 3 \end{pmatrix} + \begin{pmatrix} 4 \times 1 & 4 \times (-2) \end{pmatrix} + \begin{pmatrix} 5 \times (-1) + 1 \times 2 & 5 \times 1 + 1 \times (-3) \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 8 \end{pmatrix} + \begin{pmatrix} 4 & -8 \end{pmatrix} + \begin{pmatrix} -3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \end{pmatrix}$$

$$\boldsymbol{A}_{21}\boldsymbol{B}_{11} + \boldsymbol{A}_{22}\boldsymbol{B}_{21} + \boldsymbol{A}_{23}\boldsymbol{B}_{31} = \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \times (-1) + 2 \times 0 & 0 \times 2 + 2 \times 3 \\ 3 \times (-1) + 0 \times 0 & 3 \times 2 + 0 \times 3 \end{pmatrix} + \begin{pmatrix} 1 \times 1 & 1 \times (-2) \\ 2 \times 1 & 2 \times (-2) \end{pmatrix} + \begin{pmatrix} 1 \times (-1) + 3 \times 2 & 1 \times 1 + 3 \times (-3) \\ 0 \times (-1) + 1 \times 2 & 0 \times 1 + 1 \times (-3) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 6 \\ -3 & 6 \end{pmatrix} + \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} + \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 6 & -4 \\ 1 & -1 \end{pmatrix}$$

Conclusively

$$\begin{pmatrix} \boldsymbol{A}_{11}\boldsymbol{B}_{11} + \boldsymbol{A}_{12}\boldsymbol{B}_{21} + \boldsymbol{A}_{13}\boldsymbol{B}_{31} \\ \boldsymbol{A}_{21}\boldsymbol{B}_{11} + \boldsymbol{A}_{22}\boldsymbol{B}_{21} + \boldsymbol{A}_{23}\boldsymbol{B}_{31} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 6 & -4 \\ 1 & -1 \end{pmatrix}$$

For the confirmation, we implment multiplication of matrixes directly form the original matrixes.

$$\begin{pmatrix}1 & 2 & 4 & 5 & 1\\ 0 & 2 & 1 & 1 & 3\\ 3 & 0 & 2 & 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 2\\ 0 & 3\\ 1 & -2\\ -1 & 1\\ 2 & -3\end{pmatrix}$$

$$= \begin{pmatrix}1\times(-1)+2\times 0+4\times 1+5\times(-1)+1\times 2 & 1\times 2+2\times 3+4\times(-2)+5\times 1+1\times(-3)\\ 0\times(-1)+2\times 0+1\times 1+1\times(-1)+3\times 2 & 0\times 2+2\times 3+1\times(-2)+1\times 1+3\times(-3)\\ 3\times(-1)+0\times 0+2\times 1+0\times(-1)+1\times 2 & 3\times 2+0\times 3+2\times(-2)+0\times 1+1\times(-3)\end{pmatrix}$$

$$= \begin{pmatrix}-1+0+4-5+2 & 2+6-8+5-3\\ 0+0+1-1+6 & 0+6-2+1-9\\ -3+0+2-0+2 & 6+0-4+0-3\end{pmatrix} = \begin{pmatrix}0 & 2\\ 6 & -4\\ 1 & -1\end{pmatrix}$$

We can confirm adequacy of rules of partial matrix calculation.