V-1-2. Basic arithmetic operations of matrix

We learned how to use matrix in previous paragraph. However, we cannot use this technique without knowledge or calculation rule. Arithmetic operations of matrix are introduced in this paragraph.

1. Sum of matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{m} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1m} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nn}+b_{nm} \end{pmatrix}$$

Formula 48

- Confirmation of adequacy above rule

In the case of 2×2 matrix as example.

Calculation applying the rule is as follow

$$\begin{pmatrix} a & b \\ k & l \end{pmatrix} + \begin{pmatrix} c & d \\ m & n \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ k+m & l+n \end{pmatrix}$$

We consider the matrix is abbreviated form of matrix of vectors, by removing bases (unit vectors) . The original form is as follow.

$$\begin{pmatrix} a\boldsymbol{e_1} & b\boldsymbol{e_2} \\ k\boldsymbol{e_1} & l\boldsymbol{e_2} \end{pmatrix} + \begin{pmatrix} c\boldsymbol{e_1} & d\boldsymbol{e_2} \\ m\boldsymbol{e_1} & n\boldsymbol{e_2} \end{pmatrix} = \begin{pmatrix} (a+c)\boldsymbol{e_1} & (b+d)\boldsymbol{e_2} \\ (k+l)\boldsymbol{e_1} & (l+n)\boldsymbol{e_2} \end{pmatrix}$$

We can confirm the accuracy of above equation from figure 42.

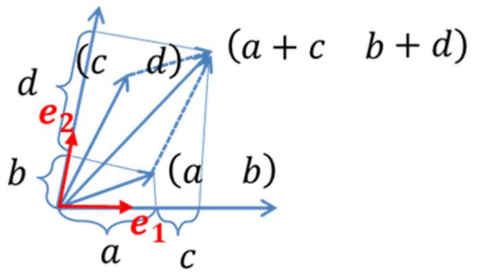

Fig.42.　Sum of vectors and matrix

The rule of abbreviation is as follow.

$$\boldsymbol{a} \leftrightarrow a\boldsymbol{e_1} \leftrightarrow (a \quad 0)$$
$$\boldsymbol{b} \leftrightarrow b\boldsymbol{e_2} \leftrightarrow (0 \quad b)$$
$$\boldsymbol{c} \leftrightarrow c\boldsymbol{e_1} \leftrightarrow (c \quad 0)$$
$$\boldsymbol{d} \leftrightarrow d\boldsymbol{e_2} \leftrightarrow (0 \quad d)$$

Then

$$(a \quad b) \leftrightarrow a\boldsymbol{e_1} + b\boldsymbol{e_2}$$
$$(c \quad d) \leftrightarrow c\boldsymbol{e_1} + d\boldsymbol{e_2}$$

$(a \quad b) + (c \quad d) = a\boldsymbol{e_1} + b\boldsymbol{e_2} + c\boldsymbol{e_1} + d\boldsymbol{e_2} = (a+c)\boldsymbol{e_1} + (b+d)\boldsymbol{e_2} = (a+c \quad b+d)$

Repeat same procedure in second line

Repeat the same procedure.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{m} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1m} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nn}+b_{nm} \end{pmatrix}$$

2. **Multiplication of scalar and matrix.**

$$\alpha \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha a_{n1} & \alpha a_{n2} & \cdots & \alpha a_{nn} \end{pmatrix}$$

α :scalar

Formula 49

- Confirmation of adequacy above rule

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{m} \end{pmatrix} = \begin{pmatrix} 2a_{11} & 2a_{12} & \cdots & 2a_{1n} \\ 2a_{21} & 2a_{22} & \cdots & 2a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 2a_{n1} & 2a_{n2} & \cdots & 2a_{nn} \end{pmatrix}$$

$$\begin{pmatrix} 2a_{11} & 2a_{12} & \cdots & 2a_{1n} \\ 2a_{21} & 2a_{22} & \cdots & 2a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 2a_{n1} & 2a_{n2} & \cdots & 2a_{nn} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{m} \end{pmatrix} = \begin{pmatrix} 3a_{11} & 3a_{12} & \cdots & 3a_{1n} \\ 3a_{21} & 3a_{22} & \cdots & 3a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 3a_{n1} & 3a_{n2} & \cdots & 3a_{nn} \end{pmatrix}$$

repeat

$$\alpha\boldsymbol{A} = \sum_{\boldsymbol{i=1}}^{\boldsymbol{\alpha}} \boldsymbol{A} = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha a_{n1} & \alpha a_{n2} & \cdots & \alpha a_{nn} \end{pmatrix}$$

3. **Multiplication of two matrixes**

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} & \cdots & a_{1l} \\ a_{21} & a_{22} & & a_{2k} & & a_{2l} \\ \vdots & & & & & \vdots \\ a_{i1} & & & a_{ik} & & a_{il} \\ \vdots & & & & & \vdots \\ a_{n1} & a_{2n} & \cdots & a_{nk} & \cdots & a_{nl} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1m} \\ b_{21} & b_{22} & & b_{2j} & & b_{2m} \\ \vdots & & & \vdots & & \vdots \\ b_{k1} & & & b_{kj} & \cdots & b_{km} \\ \vdots & & & \vdots & & \vdots \\ b_{l1} & b_{l2} & \cdots & b_{lj} & \cdots & b_{lm} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + \cdots a_{1k}b_{k1} \cdots + a_{1l}b_{l1} & \cdots & a_{11}b_{1m} + \cdots a_{1k}b_{km} \ldots a_{1l}b_{lm} \\ \vdots & a_{i1}b_{1j} + \cdots a_{ik}b_{kj} \cdots + a_{il}b_{lj} & \vdots \\ a_{n1}b_{11} + \cdots a_{nk}b_{k1} \cdots + a_{nl}b_{l1} & \cdots & a_{n1}b_{1m} + \cdots a_{nk}b_{km} \cdots + a_{nl}b_{lm} \end{pmatrix}$$

Formula 50

Procedure of calculation $\boldsymbol{AB}$ is follows

$\boldsymbol{A}$ and $\boldsymbol{B}$ are matrix

1) Multiply factor (value) at row i and column k in matrix A to factor at row k and column j in matrix B.
2) Sum the products from $k = 1$ to $k = l$
3) Factor at row i and column j in product matrix is the value obtained by step 2.

- Example

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \end{pmatrix}$$

- Confirmation of adequacy above rule

We already calculate solution of simultaneous equation applying this rule. We estimate calculation rule from the relation between two equations, on is usual expression which we learn in junior high school, the other is expression using matrix. Form this experience, we may accept the rule sensuously, though we need theoretical back ground of the rule.

Vector can express both by $1 \times l$ matrix and by unit vector as follows.

$$\boldsymbol{a} = (a_1 \quad \cdots \quad a_j \quad \cdots \quad a_l) = a_1\boldsymbol{e_1} + \cdots + a_j\boldsymbol{e_j} + \cdots + a_l\boldsymbol{e_l}$$

$$\boldsymbol{b} = (b_1 \quad \cdots \quad b_i \quad \cdots \quad b_l) = b_1\boldsymbol{e_1} + \cdots + b_i\boldsymbol{e_i} + \cdots + b_l\boldsymbol{e_l}$$

When we multiplicate $\boldsymbol{a}$ by $\boldsymbol{b}$,

$$\begin{aligned} \boldsymbol{ab} = a_1b_1\boldsymbol{e_1e_1} + \cdots + a_jb_1\boldsymbol{e_je_1} + \cdots + a_lb_1\boldsymbol{e_1e_l} \\ +a_1b_2\boldsymbol{e_1e_2} + \cdots + a_jb_2\boldsymbol{e_je_2} + \cdots + a_lb_2\boldsymbol{e_le_2} \\ \vdots \\ +a_1b_l\boldsymbol{e_1e_l} + \cdots + a_jb_l\boldsymbol{e_je_l} + \cdots + a_lb_l\boldsymbol{e_le_l} \end{aligned}$$

When $\boldsymbol{e_i} \perp \boldsymbol{e_j} \quad (i \neq j)$

$\perp$: orthogonal

$$\boldsymbol{e_ie_j} = 0$$

$$\boldsymbol{e_ie_i} = 1$$

$$\because \ \boldsymbol{e_ie_j} = |\boldsymbol{e_i}||\boldsymbol{e_j}|\cos\theta_{i-j}$$

$$\cos\left(\pm\frac{\pi}{2}\right) = 0, \quad \cos 0 = 1$$

$\boldsymbol{e_ie_j}$: inner procuct

$$\begin{aligned} \therefore \ \boldsymbol{ab} &= a_1b_1\boldsymbol{e_1e_1} + \cdots + a_ib_i\boldsymbol{e_ie_i} + \cdots + a_lb_l\boldsymbol{e_le_l} \\ &= a_1b_1 + \cdots + a_ib_i + \cdots + a_lb_l \end{aligned}$$

$$\boldsymbol{ab} = (a_1 \quad \cdots \quad a_j \quad \cdots \quad a_m)\begin{pmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{pmatrix} = a_1b_1 + \cdots + a_ib_i + \cdots + a_lb_l$$

Conclusively $\boldsymbol{ab}$ is inner product of $\boldsymbol{a}$ and $\boldsymbol{b}$

We consider $\boldsymbol{A}$ as vertical line of vectors and $\boldsymbol{B}$ as horizontal line of vectors.

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_i \\ \vdots \\ \boldsymbol{a}_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1l} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nl} \end{pmatrix}$$

$$\boldsymbol{B} = (\boldsymbol{b}_1 \quad \cdots \quad \boldsymbol{b}_j \quad \cdots \quad \boldsymbol{b}_m) = \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{l1} & \cdots & b_{lm} \end{pmatrix}$$

$$\boldsymbol{a}_i = (a_{i1} \quad \cdots \quad a_{il})$$

$$\boldsymbol{b}_j = \begin{pmatrix} b_{1j} \\ \vdots \\ b_{lj} \end{pmatrix}$$

$$\boldsymbol{AB} = \begin{pmatrix} \boldsymbol{a}_1\boldsymbol{b}_1 & \cdots & \boldsymbol{a}_1\boldsymbol{b}_j & \cdots & \boldsymbol{a}_1\boldsymbol{b}_m \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \boldsymbol{a}_i\boldsymbol{b}_1 & \cdots & \boldsymbol{a}_i\boldsymbol{b}_j & \cdots & \boldsymbol{a}_i\boldsymbol{b}_m \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \boldsymbol{a}_n\boldsymbol{b}_1 & \cdots & \boldsymbol{a}_n\boldsymbol{b}_j & \cdots & \boldsymbol{a}_n\boldsymbol{b}_m \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} & \cdots & a_{1l} \\ a_{21} & a_{22} & & a_{2k} & & a_{2l} \\ \vdots & & & & & \vdots \\ a_{i1} & & & a_{ik} & & a_{il} \\ \vdots & & & & & \vdots \\ a_{n1} & a_{2n} & \cdots & a_{nk} & \cdots & a_{nl} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1m} \\ b_{21} & b_{22} & & b_{2j} & & b_{2m} \\ \vdots & & & \vdots & & \vdots \\ b_{k1} & & & b_{kj} & \cdots & b_{km} \\ \vdots & & & \vdots & & \vdots \\ b_{l1} & b_{l2} & \cdots & b_{lj} & \cdots & b_{lm} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + \cdots a_{1k}b_{k1} \cdots + a_{1l}b_{l1} & \cdots & a_{11}b_{1m} + \cdots a_{1k}b_{km} \ldots a_{1l}b_{lm} \\ \vdots & a_{i1}b_{1j} + \cdots a_{ik}b_{kj} \cdots + a_{il}b_{lj} & \vdots \\ a_{n1}b_{11} + \cdots a_{nk}b_{k1} \cdots + a_{nl}b_{l1} & \cdots & a_{n1}b_{1m} + \cdots a_{nk}b_{km} \cdots + a_{nl}b_{lm} \end{pmatrix}$$

Some quick-minded readers may claim why we can assume $\boldsymbol{e_i} \perp \boldsymbol{e_j}$. This claim is mathematically correct. If matrix can express all vectors, they include orthogonal pairs of vectors. In such case

$$\boldsymbol{e_i e_j} \neq 0$$

However, we do not know whether $\boldsymbol{e_i}$ and $\boldsymbol{e_j}$ are orthogonal each other or they are not orthogonal each other. This is a rule of calculation. We can derive various mathematics theory by accepting the rule assuming orthogonal space. Accepting this rule, we can discuss the correlation and partial correlation and transform oblique coordinate to orthogonal coordinate.

4. Order of multiplication

Multiplication of matrixes is non-commutative as subtraction between values, though order of calculation can be changed.

$$\boldsymbol{AB} \neq \boldsymbol{BA}$$

$$(\mathbf{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$$

Formula 51

5. There is no division in matrix. Multiplication of inverse matrix is used instead of division.
6. A swapping of liens or columns makes change of sign.

- Example

Most simple case of swapping of row is as follow

$$\begin{pmatrix} a & b \\ k & l \end{pmatrix} \rightarrow \begin{pmatrix} k & l \\ a & b \end{pmatrix}$$

This swapping looks as it makes no change.

However, when we calculate the determinant.

$$\begin{vmatrix} a & b \\ k & l \end{vmatrix} = al - bk$$

$$\begin{vmatrix} k & l \\ a & b \end{vmatrix} = kb - al = -(\mathrm{al} - \mathrm{bk})$$

Then

$$\begin{vmatrix} k & l \\ a & b \end{vmatrix} = -\begin{vmatrix} a & b \\ k & l \end{vmatrix}$$

This is nature of things. In case of following simultaneous equation,

$$ax + by = \alpha$$

$$kx + ly = \beta$$

Subtraction of lower equation from upper equation is as follow.

$$(ax + by) - (kx + ly) = \alpha - \beta$$

Subtraction of upper equation from lower equation is as follow.

$$(kx + ly) - (ax + by) = \beta - \alpha$$

Swapping of upper and lower equation caused change of sign.

In the case of

$$\begin{pmatrix} a & b & c \\ k & l & m \\ s & t & u \end{pmatrix} \rightarrow \begin{pmatrix} s & t & u \\ a & b & c \\ k & l & m \end{pmatrix}$$

The process can be decomposed as follow.

$$\begin{pmatrix} a & b & c \\ k & l & m \\ s & t & u \end{pmatrix} = -\begin{pmatrix} a & b & c \\ s & t & u \\ k & l & m \end{pmatrix} = -(-\begin{pmatrix} s & t & u \\ a & b & c \\ k & l & m \end{pmatrix}) = \begin{pmatrix} s & t & u \\ a & b & c \\ k & l & m \end{pmatrix}$$

The sing changes with each swapping of row. In matrix, rules applicable to row are applicable to column.

$$\begin{pmatrix} a & b & c \\ k & l & m \\ s & t & u \end{pmatrix} = -\begin{pmatrix} a & c & b \\ s & m & l \\ k & u & t \end{pmatrix} = -(-\begin{pmatrix} c & a & b \\ m & k & l \\ u & s & t \end{pmatrix}) = \begin{pmatrix} c & a & b \\ m & k & l \\ u & s & t \end{pmatrix}$$

5. Summing and subtraction of scalar times of a row or column makes no change

- Examples

$$\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix} + \alpha \begin{pmatrix} 0 & \cdots & a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{i1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{n1} & \cdots & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1j} + \alpha a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} + \alpha a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} + a\alpha_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix} + \beta \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} + \beta a_{11} & \cdots & a_{ij} + \beta a_{1j} & \cdots & a_{in} + \beta a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & \alpha a_{nj} & \cdots & a_{nn} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix} - \frac{a_{1j}}{a_{11}} \begin{pmatrix} 0 & \cdots & a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{i1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{n1} & \cdots & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & 0 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & & \\ a_{i1} & \cdots & a_{ij} - a_{ij}\frac{a_{1j}}{a_{11}} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} - a_{nj}\frac{a_{1j}}{a_{11}} & \cdots & a_{nn} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix} + \frac{a_{i1}}{a_{11}} \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & a_{ij} - a_{ij}\frac{a_{i1}}{a_{11}} & \cdots & a_{in} - a_{in}\frac{a_{i1}}{a_{11}} \\ \vdots & \ddots & \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{nj} & & a_{nn} \end{pmatrix}$$