V-1-3. Inverse matrix and identity matrix.

Sample simultaneous equation

$$\begin{pmatrix} a & b \\ k & l \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Multiply $\frac{1}{\begin{vmatrix} a & b \\ k & l \end{vmatrix}}\begin{pmatrix} l & -b \\ -k & a \end{pmatrix}$ both side from left.

$$\frac{1}{\begin{vmatrix} a & b \\ k & l \end{vmatrix}}\begin{pmatrix} l & -b \\ -k & a \end{pmatrix}\begin{pmatrix} a & b \\ k & l \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\begin{vmatrix} a & b \\ k & l \end{vmatrix}}\begin{pmatrix} l & -b \\ -k & a \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Left side

$$\frac{1}{\begin{vmatrix} a & b \\ k & l \end{vmatrix}}\begin{pmatrix} l & -b \\ -k & a \end{pmatrix}\begin{pmatrix} a & b \\ k & l \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\begin{vmatrix} a & b \\ k & l \end{vmatrix}}(al-bk)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{(al-bk)}{(al-bk)}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\because \begin{pmatrix} l & -b \\ -k & a \end{pmatrix}\begin{pmatrix} a & b \\ k & l \end{pmatrix}$$

$$= \begin{pmatrix} al-bk & bl-bl \\ -ak+ak & -bk+al \end{pmatrix}$$

$$= \begin{pmatrix} al-bk & 0 \\ 0 & al-bk \end{pmatrix}$$

$$= (al-bk)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Right side

$$\frac{1}{\begin{vmatrix} a & b \\ k & l \end{vmatrix}}\begin{pmatrix} l & -b \\ -k & a \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\begin{vmatrix} a & b \\ k & l \end{vmatrix}}\begin{pmatrix} l\alpha - b\beta \\ -k\alpha + a\beta \end{pmatrix}$$

Left side =light side

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\begin{vmatrix} a & b \\ k & l \end{vmatrix}}\begin{pmatrix} l\alpha - b\beta \\ -k\alpha + a\beta \end{pmatrix}$$

$$x = \frac{l\alpha - b\beta}{al-bk}$$

$$y = \frac{-(k\alpha - a\beta)}{al-bk}$$

From this, we can accept $\frac{1}{\begin{vmatrix} a & b \\ k & l \end{vmatrix}}\begin{pmatrix} l & -b \\ -k & a \end{pmatrix}$ as inverse matrix of $\begin{pmatrix} a & b \\ k & l \end{pmatrix}$

$$\begin{pmatrix} a & b \\ k & l \end{pmatrix}^{-1} = \frac{1}{\begin{vmatrix} a & b \\ k & l \end{vmatrix}}\begin{pmatrix} l & -b \\ -k & a \end{pmatrix}$$

and we ca also accept $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ as identity matrix, because multiplication of $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ cause no change in following calculation.

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$$

For the confirmation we repeat the same calculation step by step.

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$=\begin{pmatrix}1\cdot x+0\cdot y\\0\cdot x+1\cdot y\end{pmatrix}$$

$$=\begin{pmatrix}x\\y\end{pmatrix}$$

The author is supposing that most of readers may understand a function of matrix, though it still fuzzy. For the confirmation, we tray same procedure in the case of 3 unknown simultaneous equation.

$$ax+by+cz=\alpha \qquad \text{i}$$

$$kx+ly+mz=\beta \qquad \text{ii}$$

$$sx+ty+uz=\gamma \qquad \text{iii}$$

i ÷ a

$$x+\frac{b}{a}y+\frac{c}{a}z=\frac{\alpha}{a} \qquad \text{i'}$$

ii ÷ k

$$x+\frac{l}{k}y+\frac{m}{k}z=\frac{\beta}{k} \qquad \text{ii'}$$

iii ÷ s

$$x+\frac{t}{s}y+\frac{u}{s}z=\frac{\gamma}{s} \qquad \text{iii'}$$

i′—ii′

$$\left(\frac{b}{a}-\frac{l}{k}\right)y+\left(\frac{c}{a}-\frac{m}{k}\right)z=\frac{\alpha}{a}-\frac{\beta}{k} \qquad \text{iv}$$

′ —③′

$$\left(\frac{b}{a}-\frac{t}{s}\right)y+\left(\frac{c}{a}-\frac{u}{s}\right)z=\frac{\alpha}{a}-\frac{\gamma}{s} \qquad \text{v}$$

$$\left(\frac{bk-a}{ak}\right)y+\left(\frac{ck-am}{ak}\right)z=\frac{k\alpha-a\beta}{ak} \qquad \text{iv'}$$

$$\left(\frac{bs-at}{as}\right)y+\left(\frac{cs-au}{as}\right)z=\frac{s\alpha-a\gamma}{as} \qquad \text{v'}$$

$$y+\left(\frac{ck-am}{bk-al}\right)z=\frac{k\alpha-a\beta}{bk-al} \qquad \text{iv''}$$

$$y+\left(\frac{cs-au}{bs-at}\right)z=\frac{s\alpha-a\gamma}{bs-at} \qquad \text{v''}$$

iv′′ − v′′

$$\left(\frac{ck-am}{bk-al}-\frac{cs-au}{bs-at}\right)z=\frac{k\alpha-a\beta}{bk-al}-\frac{s\alpha-a\gamma}{bs-a} \qquad \text{vi}$$

$$\{(ck-am)(bs-at)-(cs-au)(bk-al)\}z=(k\alpha-a\beta)(bs-at)-(s\alpha-a\gamma)(bk-al) \qquad \text{vi'}$$

$$(\cancel{bcks}-abms-ackt+a^2mt-\cancel{bcks}+abku+acls-a^2ul)z$$

$$=\cancel{bks\alpha}-abs\beta-akt\alpha+a^2t\beta-\cancel{bks\alpha}+abk\gamma+als\alpha-a^2l\gamma$$

$$a(-alu-bms-ckt+amt+bku+cls)z=a\{-(kt-ls)\alpha+(at-bs)\beta-(al-bk)\gamma\}$$

$$-a(alu+bms+ckt-amt-bku-cls)z=-a\{(kt-ls)\alpha-(at-bs)\beta+(al-bk)\gamma\}$$

$$(alu+bms+ckt-amt-bku-cls)z=(kt-ls)\alpha-(at-bs)\beta+(al-bk)\gamma$$

$$z=\frac{(kt-ls)\alpha-(at-bs)\beta+(al-bk)\gamma}{alu+bms+ckt-amt-bku-cls}$$

$$y=\frac{k\alpha-a\beta}{bk-al}-\left(\frac{ck-a}{bk-al}\right)z \qquad \text{iv'''}$$

$$y=\frac{1}{bk-al}\{(k\alpha-a\beta)-(ck-am)z\}$$

$$=\frac{1}{bk-al}\left\{\frac{(k\alpha-a\beta)(alu+bms+ckt-amt-bku-cls)-(ck-am)((kt-ls)\alpha-(at-bs)\beta+(al-bk)\gamma)}{alu+bms+ckt-amt-bku-cls}\right\}$$

About numerator

$$akl u\alpha+bkms\alpha+\cancel{ck^2t\alpha}-\cancel{akmt\alpha}-bk^2u\alpha-\cancel{ckls\alpha}-a^2lu\beta-\cancel{abms\beta}-\cancel{ackt\beta}+\cancel{a^2mt\beta}$$

$$+abku\beta+acls\beta-\cancel{ck^2t\alpha}+\cancel{akmt\alpha}+\cancel{ckls\alpha}-alms\alpha+\cancel{ackt\beta}-\cancel{a^2mt\beta}$$

$$-bcks\beta+\cancel{abms\beta}-(ck-am)(al-bk)\gamma$$

$$=al(ku-ms)\alpha-bk(ku-ms)\alpha-al(au-cs)\beta+bk(au-cs)\beta+(al-bk)(am-ck)\gamma$$

$$=(al-bk)(ku-ms)\alpha-(al-bk)(au-cs)\beta+(al-bk)(am-ck)\gamma$$

$$(al-bk)\{(ku-ms)\alpha-(au-cs)\beta+(am-ck)\gamma\}$$

Conclusion for numerator

$$y=\frac{-(ku-ms)\alpha+(au-cs)\beta-(am-ck)\gamma}{alu+bms+ckt-amt-bku-cls}$$

$$x=\frac{\alpha}{a}-\frac{b}{a}y-\frac{c}{a}z \qquad \text{i''}$$

$$x=\frac{\alpha(alu+bms+ckt-amt-bku-cls)-b(-(ku-ms)\alpha+(au-cs)\beta-(am-ck)\gamma)-c((kt-ls)\alpha-(at-\quad)\beta+(al-bk)\gamma)}{a(alu+bms+ckt-amt-bku-cls)}$$

About numerator

$$\alpha(alu+\cancel{bms}+\cancel{ckt}-amt-\cancel{bku}-\cancel{cls}+\cancel{bku}-\cancel{bms}-\cancel{ckt}+\cancel{cls})+(-abu+\cancel{bcs}+act-\cancel{bcs})\beta+(abm-\cancel{bck}-acl+\cancel{bck})\gamma$$

$$=\{(lu-mt)\alpha-(bu-ct)\beta+(bm-cl)\gamma\}$$

Conclusion for numerator

$$x = \frac{(lu - mt)\alpha - (bu - ct)\beta + (bm - cl)\gamma}{alu + bms + ckt - amt - bku - cls}$$

Summary of the result of calculation

$$x = \frac{(lu - mt)\alpha - (bu - ct)\beta + (bm - cl)\gamma}{alu + bms + ckt - amt - bku - cls} = \frac{\alpha lu + bm\gamma + c\beta t - cl\gamma - b\beta u - \alpha mt}{alu + bms + ckt - cls - bku - amt}$$

$$y = \frac{-(\mathrm{ku} - \mathrm{ms})\alpha + (\mathrm{au} - \mathrm{cs})\beta - (\mathrm{am} - \mathrm{ck})\gamma}{alu + bms + ckt - amt - bku - cls} = \frac{a\beta u + \alpha ms + kc\gamma - c\beta s - \alpha ku - am\gamma}{alu + bms + ckt - cls - bku - amt}$$

$$\mathrm{z} = \frac{(kt - ls)\alpha - (at - bs)\beta + (al - bk)\gamma}{alu + bms + ckt - amt - bku - cls} = \frac{al\gamma + b\beta s + \alpha kt - \alpha ls - bk\gamma - a\beta t}{alu + bms + ckt - cls - bku - amt}$$

When we watch this summery, we notice important phenomena.

Dominator is determinant of 3×3 matrix and coefficients of α, β, γ are determinant of 2×2 matrix. Using determinants summery is expressed as follows.

$$x = \frac{\begin{vmatrix} l & m \\ t & u \end{vmatrix}\alpha - \begin{vmatrix} b & c \\ t & u \end{vmatrix}\beta + \begin{vmatrix} b & c \\ l & m \end{vmatrix}\gamma}{\begin{vmatrix} a & b & c \\ k & l & m \\ s & t & u \end{vmatrix}}$$

$$y = \frac{-\begin{vmatrix} k & m \\ s & u \end{vmatrix}\alpha + \begin{vmatrix} a & c \\ s & u \end{vmatrix}\beta - \begin{vmatrix} a & c \\ k & m \end{vmatrix}\gamma}{\begin{vmatrix} a & b & c \\ k & l & m \\ s & t & u \end{vmatrix}}$$

$$z = \frac{\begin{vmatrix} k & l \\ s & t \end{vmatrix}\alpha - \begin{vmatrix} a & b \\ s & t \end{vmatrix}\beta + \begin{vmatrix} a & b \\ k & l \end{vmatrix}\gamma}{\begin{vmatrix} a & b & c \\ k & l & m \\ s & t & u \end{vmatrix}}$$

Remember what we did to make determinant in the case of two unknowns. Procedure was as follow.

$$\boldsymbol{A}^{-1}\boldsymbol{A}\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{\alpha}$$

$$\mathbf{X} = \boldsymbol{A}^{-1}\boldsymbol{\alpha}$$

$$\mathbf{X} = \boldsymbol{A}^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Similarly,

$$\mathbf{X} = \boldsymbol{A}^{-1}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

From this, we can presume

$$\boldsymbol{A}^{-1} = \frac{1}{\begin{vmatrix} a & b & c \\ k & l & m \\ s & t & u \end{vmatrix}} \begin{pmatrix} \begin{vmatrix} l & m \\ t & u \end{vmatrix} & -\begin{vmatrix} b & c \\ t & u \end{vmatrix} & \begin{vmatrix} b & c \\ l & m \end{vmatrix} \\ -\begin{vmatrix} k & m \\ s & u \end{vmatrix} & \begin{vmatrix} a & c \\ s & u \end{vmatrix} & -\begin{vmatrix} a & c \\ k & m \end{vmatrix} \\ \begin{vmatrix} k & l \\ s & t \end{vmatrix} & -\begin{vmatrix} a & b \\ s & t \end{vmatrix} & \begin{vmatrix} a & b \\ k & l \end{vmatrix} \end{pmatrix}$$

Confirmation of the presumption

Calculate following equation

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \frac{\begin{pmatrix} \begin{vmatrix} l & m \\ t & u \end{vmatrix} & -\begin{vmatrix} b & c \\ t & u \end{vmatrix} & \begin{vmatrix} b & c \\ l & m \end{vmatrix} \\ -\begin{vmatrix} k & m \\ s & u \end{vmatrix} & \begin{vmatrix} a & c \\ s & u \end{vmatrix} & -\begin{vmatrix} a & c \\ k & m \end{vmatrix} \\ \begin{vmatrix} k & l \\ s & t \end{vmatrix} & -\begin{vmatrix} a & b \\ s & t \end{vmatrix} & \begin{vmatrix} a & b \\ k & l \end{vmatrix} \end{pmatrix} \begin{pmatrix} a & b & c \\ k & l & m \\ s & t & u \end{pmatrix}}{\begin{vmatrix} a & b & c \\ k & l & m \\ s & t & u \end{vmatrix}}$$

$$\begin{pmatrix} \begin{vmatrix} l & m \\ t & u \end{vmatrix} & -\begin{vmatrix} b & c \\ t & u \end{vmatrix} & \begin{vmatrix} b & c \\ l & m \end{vmatrix} \\ -\begin{vmatrix} k & m \\ s & u \end{vmatrix} & \begin{vmatrix} a & c \\ s & u \end{vmatrix} & -\begin{vmatrix} a & c \\ k & m \end{vmatrix} \\ \begin{vmatrix} k & l \\ s & t \end{vmatrix} & -\begin{vmatrix} a & b \\ s & t \end{vmatrix} & \begin{vmatrix} a & b \\ k & l \end{vmatrix} \end{pmatrix} \begin{pmatrix} a & b & c \\ k & l & m \\ s & t & u \end{pmatrix}$$

$$= \begin{pmatrix} a(lu-mt)-k(bu-ct)+s(bm-cl) & -b(lu-mt)+l(bu-ct)-t(bm-cl) & c(lu-mt)-m(bu-ct)+u(bm-cl) \\ -a(ku-ms)+k(au-cs)-s(am-ck) & -b(ku-ms)+l(au-cs)-t((am-ck) & -c(ku-ms)+m(au-cs)-u(am-ck \\ a(kt-ls)-k(at-bs)+s(al-bk) & b(kt-ls)-l(at-bs)+t(al-bk) & c(kt-ls)-m(at-bs)+u(al-bk) \end{pmatrix}$$

$$= \begin{pmatrix} alu+bms+ckt-amt-bku-cls & 0 & 0 \\ 0 & alu+bms+ckt-amt-bku-cls & 0 \\ 0 & 0 & alu+bms+ckt-amt-bku-cls \end{pmatrix}$$

$$= (alu+bms+ckt-amt-bku-cls)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{vmatrix} a & b & c \\ k & l & m \\ s & t & u \end{vmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We could confirm

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Multiplying of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ makes no change. We can say, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is identity matrix, and

$$\frac{1}{\begin{vmatrix} a & b & c \\ k & l & m \\ s & t & u \end{vmatrix}} \begin{pmatrix} \begin{vmatrix} l & m \\ t & u \end{vmatrix} & -\begin{vmatrix} b & c \\ t & u \end{vmatrix} & \begin{vmatrix} b & c \\ l & m \end{vmatrix} \\ -\begin{vmatrix} k & m \\ s & u \end{vmatrix} & \begin{vmatrix} a & c \\ s & u \end{vmatrix} & -\begin{vmatrix} a & c \\ k & m \end{vmatrix} \\ \begin{vmatrix} k & l \\ s & t \end{vmatrix} & -\begin{vmatrix} a & b \\ s & t \end{vmatrix} & \begin{vmatrix} a & b \\ k & l \end{vmatrix} \end{pmatrix}$$

is inverse matrix of

$$\begin{pmatrix} a & b & c \\ k & l & m \\ s & t & u \end{pmatrix}$$

We could understand calculation of matrix in the process of solution of simultaneous equation. We can implement the procedure when we know how to make 1) determinant, 2) inverse matrix, and 3) identity matrix.

We can understand the method to make identity matrix sensuously. Identity matrix is matrix of which factors on diagonal line from upper left to lower right are 1 and the other factors are 0 as follow.

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Vector has magnitude and direction. Magnitude which has no direction is scalar. When we express moving speed as 1km/hour, this sentence does not include direction. This is scalar. When we say "go to south in 1km/hour", this sentence include direction. The word 1km/hour to south is vector. When we say "go this way straightly 100m and then turn to the left and to straight 200m", this include concept of vector. We do not know the way is south or north. However, when we fix the first direction, we can understand next direction by relative relation, right or left. Illustration of above guiding is figure 43.

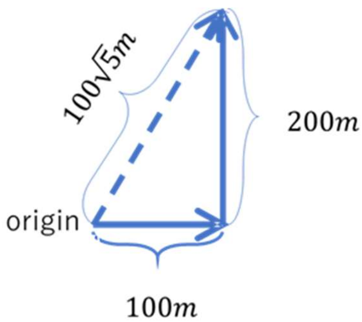

Fig. 43. Writing vectors on map

We can recognize the direction and distance of the destination by summing of the vectors. Vector $\vec{a}$ is "go straight 100m". Vector $\vec{b}$ is "turn to the left and go 200m". As a result, a new vector $(\vec{a}+\vec{b})$ "front left, distance $100\sqrt{5}$ is born. This is a combination of vector.

In the case when first direction is south east, we can define first unit vector $\overrightarrow{e_1}$ as one meter to south east and second unit vector $\overrightarrow{e_2}$ as one meter to northeast.

The relation can be expressed as follows.

$$\vec{a} = 100\overrightarrow{e_1}$$

$$\vec{b} = 200\overrightarrow{e_2}$$

We can express all vectors by product of scalar and unit vector and can produce new vector combining vectors.

$$\vec{a} + \vec{b} = 100\overrightarrow{e_1} + 200\overrightarrow{e_2}$$

Dotted line in figure 43

Vector $\vec{a}$ has no element of $\overrightarrow{e_2}$ and $\vec{b}$ has no element of $\overrightarrow{e_1}$. The relation of $\vec{a}$ and $\vec{b}$ is expressed independent each other in mathematics. We can express vectors $\vec{a}$ and $\vec{b}$ only scalar parts of element vectors.

$$\vec{a}: (100 \quad 0)$$

$$\vec{b}: (0 \quad 200)$$

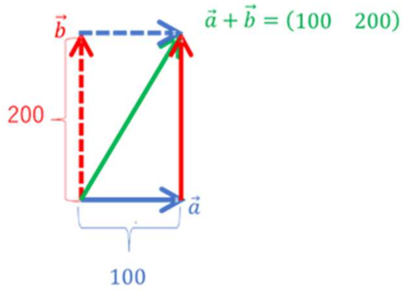

Fig. 44. Sum of vectors

We can calculate $\vec{a} + \vec{b}$ by rule of summing of matrix as follow.

$$\vec{a} + \vec{b} = (100 \quad 0) + (0 \quad 200) = (100 \quad 200)$$

We can confirm the adequacy of rule of sum of matrix from figure 44. Sometimes we can express the original vectors by aligning vectors in a column as $\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix}$.

$$\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix} = \begin{pmatrix}100\overrightarrow{e_1} & 0\overrightarrow{e_2}\\ 0\overrightarrow{e_1} & 200\overrightarrow{e_2}\end{pmatrix}$$

When we remove the unit vectors from the matrix

$$\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix} = \begin{pmatrix}100 & 0\\ 0 & 200\end{pmatrix}$$

Applying the rule of multiplication of matrix.

$$\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix} = \begin{pmatrix}100 & 0\\ 0 & 200\end{pmatrix}\begin{pmatrix}\overrightarrow{e_1}\\ \overrightarrow{e_2}\end{pmatrix} = \begin{pmatrix}100\overrightarrow{e_1}\\ 200\overrightarrow{e_2}\end{pmatrix}$$

$$\vec{a} = 100\overrightarrow{e_1}$$

$$\vec{b} = 200\overrightarrow{e_2}$$

$$\vec{a} + \vec{b} = (100 \quad 200)\begin{pmatrix}\overrightarrow{e_1} \\ \overrightarrow{e_2}\end{pmatrix} = (100\overrightarrow{e_1} \quad 200\overrightarrow{e_2})$$

From this, we can confirm the adequacy of the rule of multiplication of matrixes.

Using arrow is bothersome. We can use bold as a symbol of vector

$$\boldsymbol{a} + \boldsymbol{b} = 100\boldsymbol{e_1} + 200\boldsymbol{e_2}$$

We can understand that human being are doing synthesize of vector in their daily. In addition to this, human being is also doing transformation of vector space in their daily life. Roads are often cross not right angle particularly in local area. In the exemplifies conversation, roads cross right angle in the brain of speaker. Assuming that the roads are across having $\frac{\pi}{3}$ angle. He make his map in the brain from real map removing unnecessary information that the angle is $\frac{\pi}{3}$. Let consider the process of human being to accept real world to their strictly orthogonal world of their brain. Letting the process as matrix $\boldsymbol{A}$, vector in real world as $\boldsymbol{a} + \boldsymbol{b}$ and vector in their brain as $\boldsymbol{a}' + \boldsymbol{b}'$. Unit vectors in real world are $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$. Unit vectors in the brain are $\boldsymbol{\varepsilon_1}$ and $\boldsymbol{\varepsilon_2}$. The relations of unit vectors are as in figure 45.

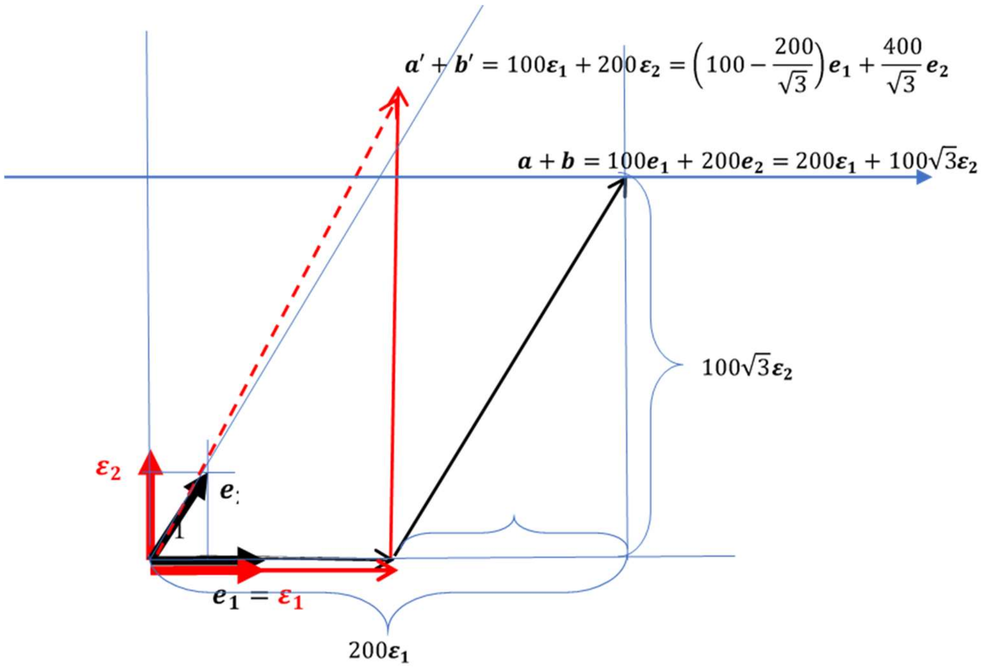

Fig. 46. Transformation of real world to the world in brain

We can express change using both sets of unit vectors.

$$\boldsymbol{a} + \boldsymbol{b} \rightarrow \boldsymbol{a}' + \boldsymbol{b}'$$

$$100\boldsymbol{e_1} + 200\boldsymbol{e_2} \rightarrow \left(100 - \frac{200}{\sqrt{3}}\right)\boldsymbol{e_1} + \frac{400}{\sqrt{3}}\boldsymbol{e_2}$$

$$200\boldsymbol{\varepsilon_1} + 100\sqrt{3}\boldsymbol{\varepsilon_2} \rightarrow 100\boldsymbol{\varepsilon_1} + 200\boldsymbol{\varepsilon_2}$$

$$\boldsymbol{a}' + \boldsymbol{b}' \rightarrow \boldsymbol{a} + \boldsymbol{b}$$

$$\left(100 - \frac{200}{\sqrt{3}}\right)e_1 + \frac{400}{\sqrt{3}}e_2 \rightarrow 100\boldsymbol{e_1} + 200\boldsymbol{e_2}$$

$$100\boldsymbol{\varepsilon_1} + 200\varepsilon_2 \rightarrow 200\boldsymbol{\varepsilon_1} + 100\sqrt{3}\boldsymbol{\varepsilon_2}$$

When we express the work of human being for the change of $\boldsymbol{a} + \boldsymbol{b} \rightarrow \boldsymbol{a}' + \boldsymbol{b}'$ as matrix $\boldsymbol{A}$, and work for change of $\boldsymbol{a}' + \boldsymbol{b}' \rightarrow \boldsymbol{a} + \boldsymbol{b}$ as matrix $\boldsymbol{B}$

$$\boldsymbol{a}' + \boldsymbol{b}' = \boldsymbol{A}(\boldsymbol{a} + \boldsymbol{b})$$

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{B}(\boldsymbol{a}' + \boldsymbol{b}')$$

When we express the vectors in the form of column of vector,

$$\boldsymbol{a} + \boldsymbol{b} = \begin{pmatrix} 100\boldsymbol{e_1} \\ 200\boldsymbol{e_2} \end{pmatrix} = \begin{pmatrix} 200\boldsymbol{\varepsilon_1} \\ 100\sqrt{3}\boldsymbol{\varepsilon_2} \end{pmatrix}$$

$$\boldsymbol{a}' + \boldsymbol{b}' = \begin{pmatrix} \left(100 - \frac{200}{\sqrt{3}}\right)\boldsymbol{e_1} \\ \frac{400}{\sqrt{3}}\boldsymbol{e_2} \end{pmatrix} = \begin{pmatrix} 100\boldsymbol{\varepsilon_1} \\ 200\boldsymbol{\varepsilon_2} \end{pmatrix}$$

In this form of expression of vector, we do not need show unit vector demonstratively, because we can identify the direction of unit vector by the position in the matrix. However, in this discussion, we need to know the origin of the number to trace the process of calculation. For this purpose, we remain the symbol of original vector as follow,

$$\boldsymbol{a} + \boldsymbol{b} = \begin{pmatrix} 100_{(e_1)} \\ 200_{(e_2)} \end{pmatrix} = \begin{pmatrix} 200_{(\epsilon_1+\varepsilon_2)} \\ 100\sqrt{3}_{(\epsilon_1+\varepsilon_2)} \end{pmatrix}$$

$$\boldsymbol{a}' + \boldsymbol{b}' = \begin{pmatrix} \left(100 - \frac{200}{\sqrt{3}}\right)_{(e_1+e_2)} \\ \frac{400}{\sqrt{3}}_{(e_1+e_2)} \end{pmatrix} = \begin{pmatrix} 100_{(\epsilon_1)} \\ 200_{(\varepsilon_2)} \end{pmatrix}$$

Then the process of the change can be expressed as follow

$$\boldsymbol{A}(\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{a}' + \boldsymbol{b}'$$

$$\boldsymbol{A}\begin{pmatrix} 100_{(e_1)} \\ 200_{(e_2)} \end{pmatrix} = \begin{pmatrix} \left(100 - \frac{200}{\sqrt{3}}\right)_{(e_1+e_2)} \\ \frac{400}{\sqrt{3}}_{(e_1+e_2)} \end{pmatrix} \qquad \text{i}$$

$$\boldsymbol{A}\begin{pmatrix} 200_{(\epsilon_1+\varepsilon_2)} \\ 100\sqrt{3}_{(\epsilon_1+\varepsilon_2)} \end{pmatrix} = \begin{pmatrix} 100_{(\epsilon_1)} \\ 200_{(\varepsilon_2)} \end{pmatrix} \qquad \text{ii}$$

$$\boldsymbol{B}(\boldsymbol{a}' + \boldsymbol{b}') = \boldsymbol{a} + \boldsymbol{b}$$

$$\boldsymbol{B}\begin{pmatrix}\left(100-\frac{200}{\sqrt{3}}\right)_{(e_1+e_2)}\\ \frac{400}{\sqrt{3}}_{(e_1+e_2)}\end{pmatrix}=\begin{pmatrix}100_{(e_1)}\\200_{(e_2)}\end{pmatrix}\qquad \text{iii}$$

Or

$$\boldsymbol{B}\begin{pmatrix}100_{(\epsilon_1)}\\200_{(\varepsilon_2)}\end{pmatrix}=\begin{pmatrix}200_{(\epsilon_1+\varepsilon_2)}\\100\sqrt{3}_{(\epsilon_1+\varepsilon_2)}\end{pmatrix}\qquad \text{iv}$$

From the equation and figure 46, we can rewrite iv as follow

$$\boldsymbol{B}\begin{pmatrix}100_{(\epsilon_1)}\\200_{(\varepsilon_2)}\end{pmatrix}=\begin{pmatrix}100_{(\epsilon_1)}+100_{(\varepsilon_2)}\\0_{(\epsilon_1)}+100\sqrt{3}_{(\varepsilon_2)}\end{pmatrix}$$

From this, we can estimate the process of calculation as follow

$$\begin{pmatrix}1 & \frac{1}{2}\\ 0 & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}100_{(\epsilon_1)}\\200_{(\varepsilon_2)}\end{pmatrix}=\begin{pmatrix}100_{(\epsilon_1)}+100_{(\varepsilon_2)}\\0_{(\epsilon_1)}+100\sqrt{3}_{(\varepsilon_2)}\end{pmatrix}=\begin{pmatrix}200\\100\sqrt{3}\end{pmatrix}$$

We can confirm adequacy of the estimation of matrix $\boldsymbol{B}$ from following equation.

$$\begin{pmatrix}1 & \frac{1}{2}\\ 0 & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}\left(100-\frac{200}{\sqrt{3}}\right)_{(e_1+e_2)}\\ \frac{400}{\sqrt{3}}_{(e_1+e_2)}\end{pmatrix}=\begin{pmatrix}100_{(e_1)}-\frac{200}{\sqrt{3}}_{(e_1+e_2)}+\frac{200}{\sqrt{3}}_{(e_1+e_2)}\\ 0_{(e_1+e_2)}+200_{(e_2)}\end{pmatrix}=\begin{pmatrix}100\\200\end{pmatrix}$$

From this, we can estimate

$$\boldsymbol{B}=\begin{pmatrix}1 & \frac{1}{2}\\ 0 & \frac{\sqrt{3}}{2}\end{pmatrix}$$

Similarly,

$$\begin{pmatrix}1 & -\frac{1}{\sqrt{3}}\\ 0 & \frac{2}{\sqrt{3}}\end{pmatrix}\begin{pmatrix}200_{(\epsilon_1+\varepsilon_2)}\\100\sqrt{3}_{(\epsilon_1+\varepsilon_2)}\end{pmatrix}=\begin{pmatrix}200_{(\epsilon_1)}-100_{(\epsilon_1+\varepsilon_2)}\\0_{(\epsilon_1+\varepsilon_2)}+200_{(\epsilon_1+\varepsilon_2)}\end{pmatrix}=\begin{pmatrix}100\\200\end{pmatrix}$$

$$\begin{pmatrix}1 & -\frac{1}{\sqrt{3}}\\ 0 & \frac{2}{\sqrt{3}}\end{pmatrix}\begin{pmatrix}100_{(e_1)}\\200_{(e_2)}\end{pmatrix}=\begin{pmatrix}100_{(e_1)}-\frac{200}{\sqrt{3}}_{(e_2)}\\ \frac{400}{\sqrt{3}}_{(e_2)}\end{pmatrix}=\begin{pmatrix}100-\frac{200}{\sqrt{3}}\\ \frac{400}{\sqrt{3}}\end{pmatrix}$$

$$\boldsymbol{A}=\begin{pmatrix}1 & -\frac{1}{\sqrt{3}}\\ 0 & \frac{2}{\sqrt{3}}\end{pmatrix}$$

For the confirmation,

$$\boldsymbol{AB} = \begin{pmatrix} 1 & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 1\times 1 & 1\times\frac{1}{2}-\frac{1}{\sqrt{3}}\times\frac{\sqrt{3}}{2} \\ 0\times 1+\frac{2}{\sqrt{3}}\times 0 & 0\times\frac{1}{2}+\frac{2}{\sqrt{3}}\times\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boldsymbol{B} = \boldsymbol{A}^{-1}$$

$$\boldsymbol{A} = \boldsymbol{B}^{-1}$$

$\boldsymbol{A}$ and $\boldsymbol{B}$ are inverse matrix each other and we can confirm that $\boldsymbol{B}$ can completely compensate the function of $\boldsymbol{A}$.

We are making skewed world in our brain by projecting real world in our brain by the function of $\boldsymbol{A}$. Inversely, we can expect to reconstruct real world by function of $\boldsymbol{A}^{-1}$. Applying similar technology, we can project flat image even on curved surface. Inversely, some may expect to reconstruct real world by function of $\boldsymbol{A}^{-1}$ to know correct world. This is not correct. Human being recognizes the world by projecting image of real world in their brain. The procedure itself is work of human being. The image of the world is not equal among human beings. The relation of brain and real world cannot be observed our inner world. So, the world cannot be correct. What we learn here is a skill to observe and recognize the relation among various world.

An important finding by above trial to make matrix as a function is nature of matrix as set of vectors. The author made matrixes by lining up vectors in row or column without any explanation. This is because of authors expectation to readers to learn such technique naturally. More ideologically, readers can accept that matrix has magnitude and directions. When we draw the arrows of vectors, the direction and length of the arrows deformed by multiplication of matrix. It is enlarged in some directions and diminished in the other directions. From this, we can say, matrix has magnitude and directionality.