V-1-4. Determinant

We already used determinant in previous calculations without theoretical explanation and calculation method of determinant. The author enforced readers to accept determinant as something magnitude or size of matrix blindly, and to join the calculation of matrix act without enough knowledge. However, readers could find inverse matrix and identity matrix already by themselves. Inverse matrix is obtained finally dividing by determinant and identity matrix is obtained by dividing determinant. From this, readers may already notice that determinant is something like ratio for standardization. This is our preliminary guess.

In later part of this paragraph, we confirm accuracy of our guess.

We consider transformation of triangles in a plane.

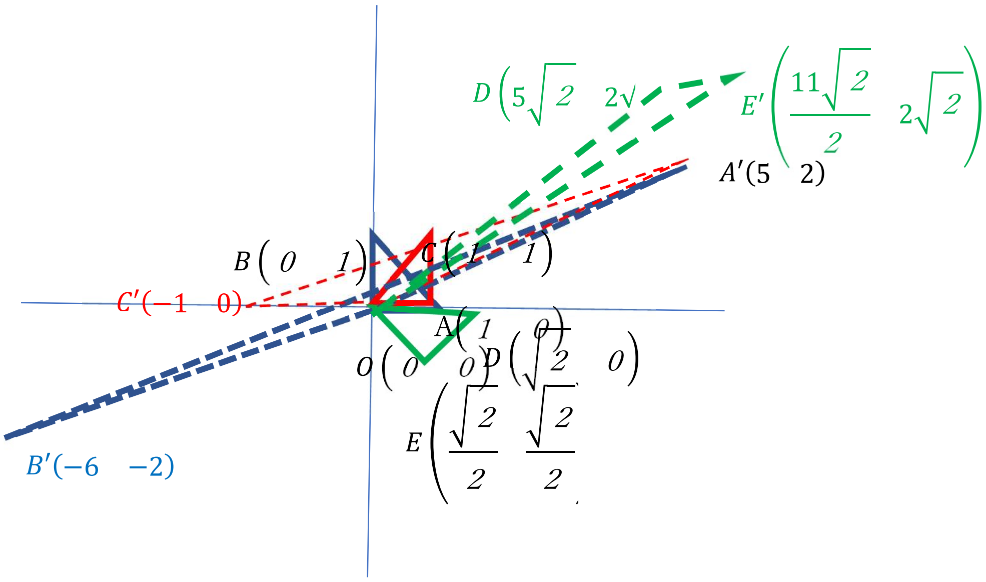


Fig.47 transformation of triangles by a matrix.

Following matrix is used for transformation.

$$\begin{pmatrix} 5 & -6 \\ 2 & -2 \end{pmatrix}$$

We transform blue, solid read and green triangles in figure 47.

Blue triangle is BOA, Red triangle is OAC, and green triangle is 0ED Dotted triangles are transformed triangle from solid triangles in each color.

Coordinates of vertex

$$\text{O}: (0 \quad 0)$$

$$\text{A}: (1 \quad 0)$$

$$\text{B}: (0 \quad 1)$$

$$\text{C}: (1 \quad 1)$$

$$\mathrm{D}:\begin{pmatrix}\sqrt{2} & 0\end{pmatrix}$$

$$\mathrm{E}:\begin{pmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{pmatrix}$$

Each vertex moves to new coordinate by transformation

$$\mathrm{A}':\begin{pmatrix}5 & -6\\ 2 & -2\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}=\begin{pmatrix}5\\ 2\end{pmatrix}$$

$$\mathrm{B}':\begin{pmatrix}5 & -6\\ 2 & -2\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}=\begin{pmatrix}-6\\ -2\end{pmatrix}$$

$$\mathrm{C}':\begin{pmatrix}5 & -6\\ 2 & -2\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}=\begin{pmatrix}-1\\ 0\end{pmatrix}$$

$$\mathrm{D}':\begin{pmatrix}5 & -6\\ 2 & -2\end{pmatrix}\begin{pmatrix}\sqrt{2}\\ 0\end{pmatrix}=\begin{pmatrix}5\sqrt{2}\\ 2\sqrt{2}\end{pmatrix}$$

$$\mathrm{E}':\begin{pmatrix}5 & -6\\ 2 & -2\end{pmatrix}\begin{pmatrix}\frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2}\end{pmatrix}=\begin{pmatrix}\frac{11}{2}\sqrt{2}\\ 2\sqrt{2}\end{pmatrix}$$

We calculate area of triangles.

$$\varDelta\mathrm{BOA}=\varDelta\mathrm{OAC}=\varDelta\ 0\mathrm{ED}=\frac{1}{2}\times 1\times 1=\frac{1}{2}$$

$$\Delta\mathrm{B'OA'}=\frac{1}{2}\times\mathrm{OH}\times 2+\frac{1}{2}\times\mathrm{OH}\times 2=2\times\frac{1}{2}=1$$

$$\Delta\mathrm{OA'C'}=\frac{1}{2}\times\mathrm{OC'}\times 2=1$$

$$\Delta 0\mathrm{E'D'}=\frac{1}{2}\times\mathrm{E'D'}\times 2\sqrt{2}=\frac{1}{2}\times\frac{\sqrt{2}}{2}\times 2\sqrt{2}=1$$

$$\begin{vmatrix}5 & -6\\ 2 & -2\end{vmatrix}=2$$

$$\frac{\Delta\mathrm{B'OA'}}{\varDelta\mathrm{BOA}}=\frac{\Delta\mathrm{OA'C'}}{\varDelta\mathrm{OAC}}=\frac{\Delta 0\mathrm{E'D'}}{\varDelta\ 0\mathrm{ED}}=\begin{vmatrix}5 & -6\\ 2 & -2\end{vmatrix}=2$$

We could confirm the expansion ratio of each triangle is the same as determinant.

We can confirm this fact in 3dimension space. We consider transformation of a unit delta cone by matrix $\boldsymbol{A}$

$$\boldsymbol{A}=\begin{pmatrix}1 & -5 & -4\\ 2 & 4 & 3\\ 2 & -1 & -1\end{pmatrix}$$

$$\begin{vmatrix}1 & -5 & -4\\ 2 & 4 & 3\\ 2 & -1 & -1\end{vmatrix}=-4-30+8+32-10+3=-1$$

Unit cone

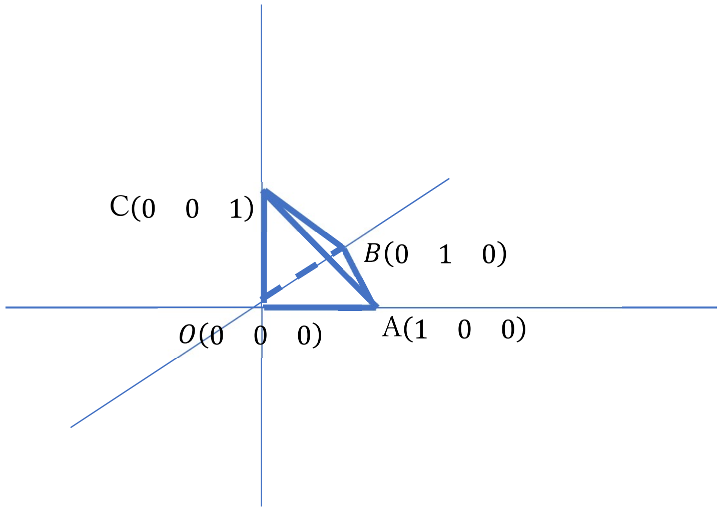


Fig. 48 Unit delta cone for transformation

$$V_o: volume\ of\ delta\ cone\ \mathrm{OABC}$$

$$V_o = \frac{1}{3} \times \frac{1}{2} \times 1 \times 1 \times 1 = \frac{1}{6}$$

Transformation

$$O \begin{pmatrix} 1 & -5 & -4 \\ 2 & 4 & 3 \\ 2 & -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A' \begin{pmatrix} 1 & -5 & -4 \\ 2 & 4 & 3 \\ 2 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$B' \begin{pmatrix} 1 & -5 & -4 \\ 2 & 4 & 3 \\ 2 & -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ -1 \end{pmatrix}$$

$$C' \begin{pmatrix} 1 & -5 & -4 \\ 2 & 4 & 3 \\ 2 & -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ -1 \end{pmatrix}$$

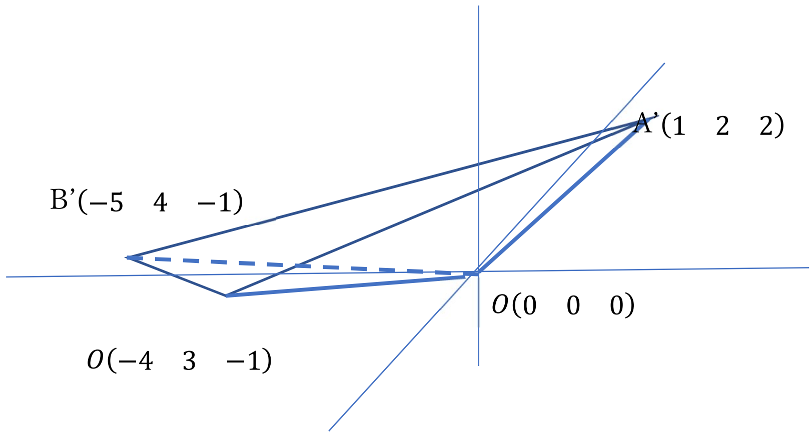


Fig.49. Transformed delta cone.

About $\Delta A'B'C'$

The plane including $\Delta A'B'C'$

$$\overleftarrow{B'A'} = (-5 \quad 4 \quad -1) - (1 \quad 2 \quad 2) = (-6 \quad 2 \quad -3)$$

$$\overleftarrow{C'A'} = (-4 \quad 3 \quad -1) - (1 \quad 2 \quad 2) = (-5 \quad 1 \quad -3)$$

The plane including $\overleftarrow{B'A'}$ and $\overleftarrow{C'A'}$

$$u\overleftarrow{B'A'} + \overleftarrow{C'A'} + \overleftarrow{A'O} = u(-6 \quad 2 \quad -3) + v(-5 \quad 1 \quad -3) + (1 \quad 2 \quad 2)$$

$$= (-6u - 5v + 1 \quad 2u + v + 2 \quad -3u - 3v + 2)$$

$$-6u - 5v + 1 = x \qquad \text{i}$$

$$2u + v + 2 = y \qquad \text{ii}$$

$$-3u - 3v + 2 = z \qquad \text{iii}$$

$\frac{1}{3}$iii

$$-u - v + \frac{2}{3} = \frac{1}{3}z \qquad \text{iii'}$$

ii + iii′

$$u + 2 + \frac{2}{3} = y + \frac{1}{3}z$$

$$u = y + \frac{1}{3}z - \frac{8}{3} \qquad \text{iv}$$

i − 5 × iii′

$$-u + 1 - \frac{10}{3} = x - \frac{5}{3}z$$

$$u = -x + \frac{5}{3}z - \frac{7}{3} \qquad \text{v}$$

iv = v

$$y + \frac{1}{3}z - \frac{8}{3} = -x + \frac{5}{3}z - \frac{7}{3}$$

$$x + y - \frac{4}{3}z - \frac{1}{3} = 0$$

$$3x + 3y - 4z - 1 = 0$$

Distance from the plane to O

From the rule of distance from the plate*

$$\left|\overleftarrow{OH}\right| = \frac{3 \times 0 + 3 \times 0 - 4 \times 0 - 1}{\sqrt{3^2 + 3^2 + (-4)^2}} = \frac{|-1|}{\sqrt{34}}$$

- * Rule of distance of a point from the plate

  Plate

$$ax + by + cz + d = 0$$

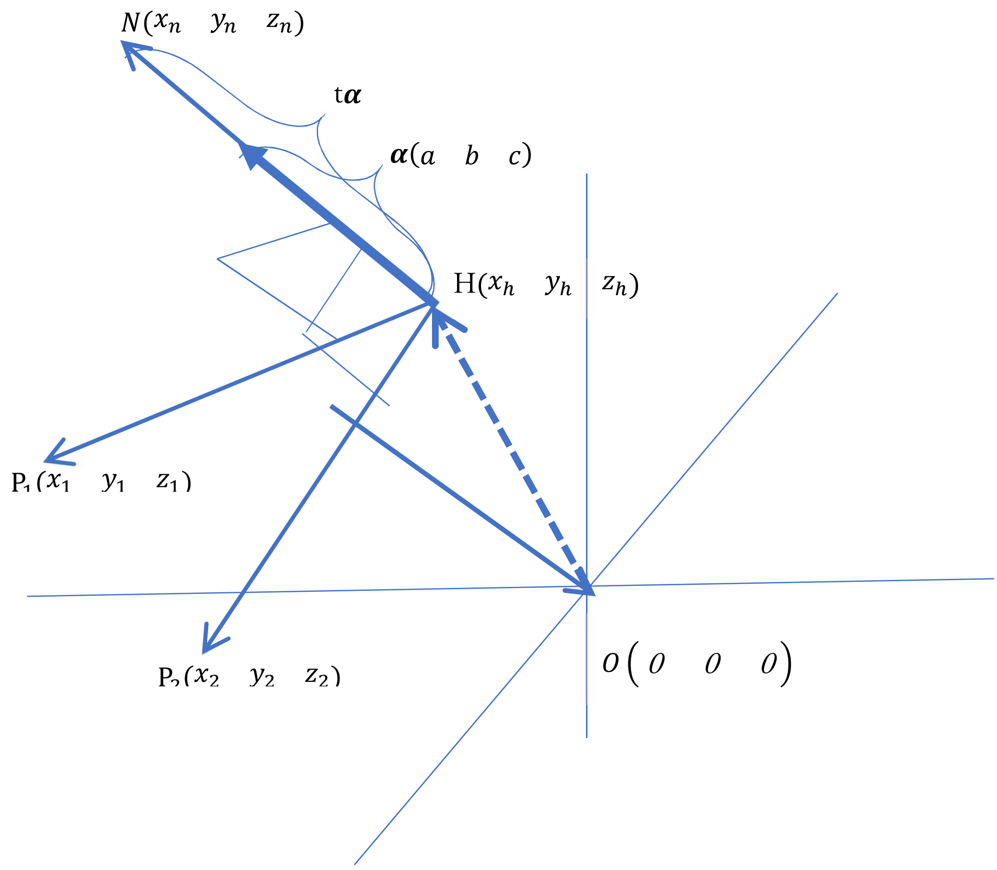
  Point

$$(x_n \quad y_n \quad z_n)$$

Distance

$$|\overleftarrow{\mathrm{NH}}| = \frac{|ax_n + by_n + cz_n + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Proof



Let N as a point on a normal line of a plane. Point on the plane is denoted as $(x \quad y \quad z)$. The foot of the normal line (H) is $(x_h \quad y_h \quad z_h)$ and arrow head of the vector $\overleftarrow{\mathrm{NH}}$ is $(x_n \quad y_n \quad z_n)$

When we assume an element vector on NH as $(a \quad b \quad c)$ and express by $\boldsymbol{\alpha}$

$$\overleftarrow{\mathrm{NH}} = t\boldsymbol{\alpha}$$

$$\boldsymbol{e} = (a \quad b \quad c)$$

$$\overleftarrow{\mathrm{NH}} = (x_n - x_h \quad y_n - y_h \quad z_n - z_h) = t(a \quad b \quad c) \qquad \text{i} \qquad\qquad \text{i}$$

Vector on the plane ($\boldsymbol{p}$) is

$$\boldsymbol{p} = (x - x_h \quad y - y_h \quad z - z_h)$$

Vectors $\overleftarrow{\mathrm{NH}}$ and $\boldsymbol{p}$ are orthogonal each other and their inner product is 0.

$$\overleftarrow{\mathrm{NH}} \cdot \boldsymbol{p} = 0$$

$$(x_n - x_h \quad y_n - y_h \quad z_n - z_h)\begin{pmatrix} x - x_h \\ y - y_h \\ z - z_h \end{pmatrix}$$

$$= t(a \quad b \quad c)\begin{pmatrix} x - x_h \\ y - y_h \\ z - z_h \end{pmatrix}$$

$$= t\big(a(x - x_h) + b(y - y_h) + c(z - z_h)\big) = 0$$

$$t \neq 0$$

$$ax + by + cz - (ax_h + by_h + cz_h) = 0$$

Here

$$ax_h + by_h + cz_h + d = 0$$

Then

$$ax + by + cz + d = 0$$

Multiply $(a \quad b \quad c)^T$ to the both side of equation i from rite side

$$(x_n - x_h \quad y_n - y_h \quad z_n - z_h)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = t(a \quad b \quad c)\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$ax_n - ax_h + by_n - by_h + cz_n - cz_h = t(a^2 + b^c + c^2)$$

$$ax_n + by_n + cz_n - (ax_h + by_h + cz_h) = t(a^2 + b^c + c^2)$$

$$ax_h + by_h + cz_h = -d$$

$$a^2 + b^c + c^2 \neq 0$$

$$t = \frac{ax_n + by_n + cz_n + d}{a^2 + b^c + c^2}$$

$$\left|\overleftarrow{\mathrm{NH}}\right| = t|\boldsymbol{\alpha}| = t\sqrt{a^2 + b^c + c^2} = \frac{ax_n + by_n + cz_n + d}{a^2 + b^c + c^2}\sqrt{a^2 + b^c + c^2} = \frac{ax_0 + by_0 + cz_o + d}{\sqrt{a^2 + b^c + c^2}}$$

Q.E.D

Area of $\Delta \mathrm{A'B'C'}$

Angle $\angle \mathrm{B'A'C'}$

$$\angle \mathrm{B'A'C'} = \theta$$

Inner product of $\overleftarrow{\mathrm{B'A'}}$ and $\overleftarrow{\mathrm{C'A'}}$

$$\overleftarrow{\mathrm{B'A'}} = (-6 \quad 2 \quad -3)$$

$$\overleftarrow{\mathrm{C'A'}} = (-5 \quad 1 \quad -3)$$

$$\left|\overleftarrow{\mathrm{B'A'}}\right| = \sqrt{(-6)^2 + 2^2 + (-3)^2} = \sqrt{49} = 7$$

$$\left|\overleftarrow{\mathrm{C'A'}}\right| = \sqrt{(-5)^2 + 1^2 + (-3)^2} = \sqrt{35}$$

$$\overleftarrow{\mathrm{B'A'}} \cdot \overleftarrow{\mathrm{C'A'}} = (-6) \times (-5) + 2 \times 1 + (-3) \times (-3) = 41$$

$$\left|\overleftarrow{\mathrm{B'A'}}\right|\left|\overleftarrow{\mathrm{C'A'}}\right| \cos\theta = \overleftarrow{\mathrm{B'A'}} \cdot \overleftarrow{\mathrm{C'A'}}$$

$$\cos\theta = \frac{\overleftarrow{B'A'} \cdot \overleftarrow{C'A'}}{\left|\overleftarrow{B'A'}\right|\left|\overleftarrow{C'A'}\right|}$$

$$\sin^2\theta = 1 - \cos^2\theta = \frac{\left(\overleftarrow{B'A'} \cdot \overleftarrow{C'A'}\right)^2}{\left|\overleftarrow{B'A'}\right|^2\left|\overleftarrow{C'A'}\right|^2} = \frac{\left|\overleftarrow{B'A'}\right|^2\left|\overleftarrow{C'A'}\right|^2 - \left(\overleftarrow{B'A'} \cdot \overleftarrow{C'A'}\right)^2}{\left|\overleftarrow{B'A'}\right|^2\left|\overleftarrow{C'A'}\right|^2}$$

$$\sin\theta = \frac{\sqrt{\left|\overleftarrow{B'A'}\right|^2\left|\overleftarrow{C'A'}\right|^2 - \left(\overleftarrow{B'A'} \cdot \overleftarrow{C'A'}\right)^2}}{\left|\overleftarrow{B'A'}\right|\left|\overleftarrow{C'A'}\right|}$$

S: area of $\Delta A'B'C'$

$$S = \frac{1}{2}\left|\overleftarrow{B'A'}\right|\left|\overleftarrow{C'A'}\right|\sin\theta = \frac{1}{2}\sqrt{\left|\overleftarrow{B'A'}\right|^2\left|\overleftarrow{C'A'}\right|^2 - \left(\overleftarrow{B'A'} \cdot \overleftarrow{C'A'}\right)^2}$$

$$(\because \text{ area of triangle: } \frac{1}{2} \times base \times height)$$

$$= \frac{1}{2}\sqrt{49 \times 35 - 41^2} = \frac{1}{2}\sqrt{1715 - 1681} = \frac{1}{2}\sqrt{34}$$

V_t: volume of delta cone $OA'B'C'$

$$V_t = \frac{1}{3} \times S \times height(\,distance\ btween\ o\ and\ the\ plane)$$

$$= \frac{1}{3} \times \frac{1}{2} \times \sqrt{34} \times \frac{|-1|}{\sqrt{34}} = \frac{1}{6}|-1|$$

$$\frac{V_t}{V_o} = \frac{\frac{1}{6}|-1|}{\frac{1}{6}} = \begin{vmatrix} 1 & -5 & -4 \\ 2 & 4 & 3 \\ 2 & -1 & -1 \end{vmatrix} = -1$$

We could confirm that determinant is ratio of volume before and after the transformation.