## V-1-5. Surrus' s rule

Surrus's rule is introduced as a calculation method of determinant of matrix in many text books. The name of Surrus's rule is "Sarasu no houhou" in Japanese. The meaning of "houhou" is method. The author learned Surrus's rule as a convenient calculation tool for determinant at first. The author is thinking that the introduction of Surrus's rule should not be a simple introduction of convenient calculation method, and that it should be theoretical explanation of space geometrical structure. Because, the process of calculation is too long to finish without any mistakes, when we use Surrus's rule in calculation of determinant of large matrix. Row reduction method is recommendable as easy and robust calculation method for large matrix. Moreover, Surrus's rule includes various important concept of in linear algebra, and it is base of cofactor expansion.

However, theoretical explanation of Surrus's rule is technically difficult, because of limitation of illustration of multidimensional space in 2-dimensional plane. The author selects a method expansion of concept illustrated in 2-dimensional image to multidimensional space by analogy. Figure 50 is an explanation of transformation of unit square by a $2 \times 2$ matrix.

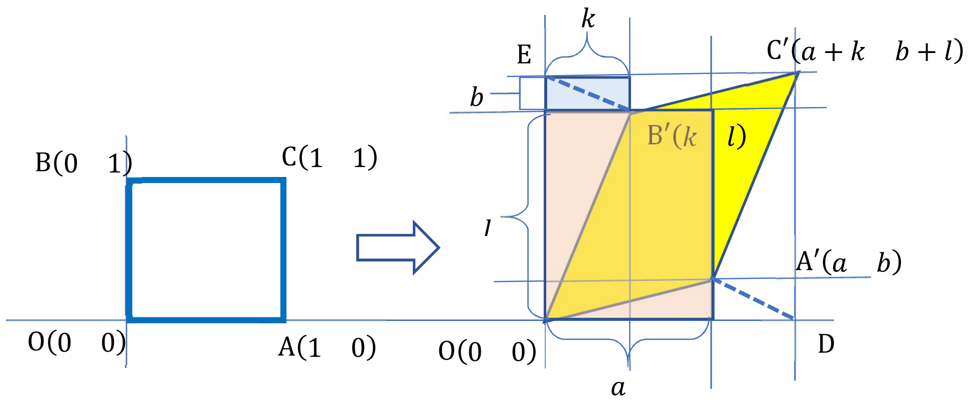



Fig. 50. Transformation of unit square by a matrix

The matrix is $\begin{pmatrix} a & b \\ k & l \end{pmatrix}$

Yellow parallelogram is made by the multiplication of the matrix. The determinant of the matrix is as follow

$$\begin{vmatrix} a & b \\ k & l \end{vmatrix} = al - bk$$

As in the explanation of previous paragraph, the determinant is the ratio of area of yellow parallelogram and unit square.

$$\frac{V_t}{V_u} = \begin{vmatrix} a & b \\ k & l \end{vmatrix} = al - bk$$

$V_t$: area of yelow parallelogramOB′A′

$V_u$: area of unit square

Here,

$$V_u = 1$$

Then

$$V_t = al - bk$$

On the other hand, when we calculate the area of yellow parallelogram geometrically.

$$\text{Area of square OEC'D} = \text{area of yellow parallelogram OB'C'A'} + \Delta\text{OEB'} + \Delta\text{EC'B'} + \Delta\text{DA'C'} + \Delta\text{OA'D}$$

$$\text{Area of square OEC'D} = (a + k)(b + l)$$

$$\Delta\text{OEB'} = \Delta\text{DA'C'} = \frac{1}{2}k(b + l)$$

$$\Delta\text{EC'B'} = \Delta\text{OA'D} = \frac{1}{2}b(a + k)$$

$$\text{Area of square OEC'D} = V_t + \Delta\text{OEB'} + \Delta\text{EC'B'} + \Delta\text{DA'C'} + \Delta\text{OA'D}$$

$$V_t = \text{Area of square OEC'D} - (\Delta\text{OEB'} + \Delta\text{EC'B'} + \Delta\text{DA'C'} + \Delta\text{OA'D})$$

$$= (a + k)(b + l) - k(b + l) - b(a + k)$$

$$= ak + al + kb + kl - kb - kl - ab - kb$$

$$= al - kb$$

We can confirm the area of area of yelow parallelogramOB′A′ ($V_t$) is determinant. We already know determinant is expansion ratio by the matrix and area of original unit square is 1. This is not sensational. However, when we watch the illustration carefully, we can notice following important fact.

Value of $al$ is area of orange rectangle and value of $kb$ is area of light blue rectangle. The area of the parallelogram can be obtained as a difference between the two rectangles. Moreover, the relation of horizontal and vertical are reverse between two rectangles. If we express this relation as positive and negative. We can accept the calculation not as difference but as sum of values in reverse direction.

$$V_t = al + (-kb)$$

This is Surrus's rule. We can expand this method to multidimensional space. When we remove a dimension from n-dimensional space. We can calculate (n-1)-dimensional super-volume (magnitude). We can accept the magnitude as n-dimensional super area. This is the determinant of matrix which not include the dimension as an element. We call the matrix as cofactor matrix. There are $n$ set of dimension and cofactor matrix. The magnitude of total n-dimensional super volume is total sum of products of dimension and its cofactor including sign. Figure 51 is schematic expression of Surrus's rule in multidimension.

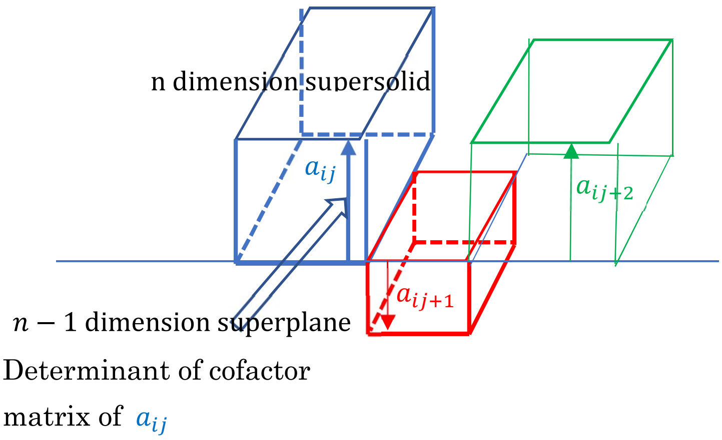



Fig. 51. Schematic illustration of Surrus's rule

The rule is mathematically explained as follows.

At first we select first column as a dimension to remove from original matrix.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11}+0+\cdots+0 & a_{12} & \cdots & a_{1n} \\ a_{21}+0+\cdots+0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+0+\cdots+0 & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11}+0+\cdots+0 & a_{12} & \cdots & a_{1n} \\ 0+a_{21}+\cdots+0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0+\cdots+0+a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \frac{1}{n}\begin{vmatrix} na_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \frac{1}{n}\begin{vmatrix} 0 & a_{12} & \cdots & a_{1n} \\ na_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \cdots + \frac{1}{n}\begin{vmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ na_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Swapping of rows

$$= \frac{(-1)^0}{n}\begin{vmatrix} na_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \frac{(-1)^1}{n}\begin{vmatrix} na_{21} & a_{22} & \cdots & a_{2n} \\ 0 & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \cdots + \frac{(-1)^{n-1}}{n}\begin{vmatrix} na_{n1} & a_{n2} & \cdots & a_{nn} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n-1\,2} & \cdots & a_{n-1\,n} \end{vmatrix}$$

Subtraction of scalar times of a row from other rows

$$= \frac{(-1)^0}{n}\begin{vmatrix} na_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \frac{(-1)^1}{n}\begin{vmatrix} na_{21} & 0 & \cdots & 0 \\ 0 & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \cdots + \frac{(-1)^{n-1}}{n}\begin{vmatrix} na_{n1} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n-1\,2} & \cdots & a_{n-1\,n} \end{vmatrix}$$

$$= \frac{(-1)^0}{n}\begin{vmatrix} na_{11} & 0 \\ 0 & \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} \end{vmatrix} + \frac{(-1)^1}{n}\begin{vmatrix} na_{21} & 0 \\ 0 & \begin{vmatrix} a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} \end{vmatrix} + \cdots + \frac{(-1)^{n-1}}{n}\begin{vmatrix} na_{n1} & 0 \\ 0 & \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n-1\,2} & \cdots & a_{n-1\,n} \end{vmatrix} \end{vmatrix}$$

$$= (-1)^0 a_{11} \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} + (-1)^1 a_{21} \begin{vmatrix} a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} + \cdots + (-1)^{n-1} a_{n1} \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n-1\,2} & \cdots & a_{n-1\,n} \end{vmatrix}$$

This is basic procedure. The name of this procedure is cofactor expansion. We can reduce one dimension by this procedure and can obtain $n$ determinants of $(n-1)$-dimensional matrix. We can repeat same procedure to reach 1 dimensional matrix. It is bothersome to write the process downward from n-dimension to 1-dimension. The author explains the process upward from 2-dimension.

Case 1 ( $2 \times 2$)

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11}+0 & a_{12} \\ a_{21}+0 & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11}+0 & a_{12} \\ 0+a_{21} & a_{22} \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 2a_{11} & a_{12} \\ 0 & a_{22} \end{vmatrix} + \frac{1}{2}\begin{vmatrix} 0 & a_{12} \\ 2a_{21} & a_{22} \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 2a_{11} & a_{12} \\ 0 & a_{22} \end{vmatrix} - \frac{1}{2}\begin{vmatrix} 2a_{21} & a_{22} \\ 0 & a_{12} \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 2a_{11} & 0 \\ 0 & a_{22} \end{vmatrix} - \frac{1}{2}\begin{vmatrix} 2a_{21} & 0 \\ 0 & a_{12} \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 2a_{11} & 0 \\ 0 & |a_{22}| \end{vmatrix} - \frac{1}{2}\begin{vmatrix} 2a_{21} & a_{22} \\ 0 & |a_{12}| \end{vmatrix} = \begin{vmatrix} a_{11} & 0 \\ 0 & |a_{22}| \end{vmatrix} - \begin{vmatrix} a_{21} & 0 \\ 0 & |a_{12}| \end{vmatrix}$$

$$= a_{11}|a_{22}| - a_{21}|a_{12}| = a_{11}a_{22} - a_{21a_{12}}$$

Case 2 ( $3 \times 3$)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}+0+0 & a_{12} & a_{13} \\ a_{21}+0+0 & a_{22} & a_{23} \\ a_{31}+0+0 & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}+0+0 & a_{12} & a_{13} \\ 0+a_{21}+0 & a_{22} & a_{23} \\ 0+0+a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \frac{1}{3}\begin{vmatrix} 3a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \frac{1}{3}\begin{vmatrix} 0 & a_{12} & a_{13} \\ 3a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \frac{1}{3}\begin{vmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 3a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \frac{1}{3}\begin{vmatrix} 3a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \frac{-1}{3}\begin{vmatrix} 3a_{21} & a_{22} & a_{23} \\ 0 & a_{12} & a_{13} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \frac{1}{3}\begin{vmatrix} 3a_{31} & a_{32} & a_{33} \\ 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \end{vmatrix}$$

$$= \frac{1}{3}\begin{vmatrix} 3a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \frac{-1}{3}\begin{vmatrix} 3a_{21} & 0 & 0 \\ 0 & a_{12} & a_{13} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \frac{1}{3}\begin{vmatrix} 3a_{31} & 0 & 0 \\ 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \end{vmatrix}$$

$$= \frac{1}{3}\begin{vmatrix} 3a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \frac{-1}{3}\begin{vmatrix} 3a_{21} & 0 & 0 \\ 0 & a_{12} & a_{13} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \frac{1}{3}\begin{vmatrix} 3a_{31} & 0 & 0 \\ 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \end{vmatrix}$$

$$= \frac{1}{3}\begin{vmatrix} 3a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \frac{-1}{3}\begin{vmatrix} 3a_{21} & 0 & 0 \\ 0 & a_{12} & a_{13} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \frac{1}{3}\begin{vmatrix} 3a_{31} & 0 & 0 \\ 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

Repeat same procedure as in case 1 $(2 \times 2)$

$$= a_{11}\left(\begin{vmatrix} a_{22} & 0 \\ 0 & a_{33} \end{vmatrix} - \begin{vmatrix} a_{23} & 0 \\ 0 & a_{32} \end{vmatrix}\right) - a_{21}\left(\begin{vmatrix} a_{12} & 0 \\ 0 & a_{33} \end{vmatrix} - \begin{vmatrix} a_{13} & 0 \\ 0 & a_{32} \end{vmatrix}\right) + a_{31}\left(\begin{vmatrix} a_{12} & 0 \\ 0 & a_{23} \end{vmatrix} - \begin{vmatrix} a_{13} & 0 \\ 0 & a_{22} \end{vmatrix}\right)$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - (a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32})$$

Case 3 ( $4 \times 4$)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{vmatrix} a_{11}+0+0+0 & a_{12} & a_{13} & a_{14} \\ 0+a_{21}+0+0 & a_{22} & a_{23} & a_{24} \\ 0+0+a_{31}+0 & a_{32} & a_{33} & a_{34} \\ 0+0+0+a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$= \frac{1}{4}\begin{vmatrix} 4a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{vmatrix} + \frac{1}{4}\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 4a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{vmatrix} + \frac{1}{4}\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 4a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{vmatrix} + \frac{1}{4}\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 4a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$= \frac{1}{4}\begin{vmatrix} 4a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{vmatrix} - \frac{1}{4}\begin{vmatrix} 4a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & a_{12} & a_{13} & a_{14} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{vmatrix} + \frac{1}{4}\begin{vmatrix} 4a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{42} & a_{43} & a_{44} \end{vmatrix} - \frac{1}{4}\begin{vmatrix} 4a_{41} & a_{42} & a_{43} & a_{44} \\ 0 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \end{vmatrix}$$

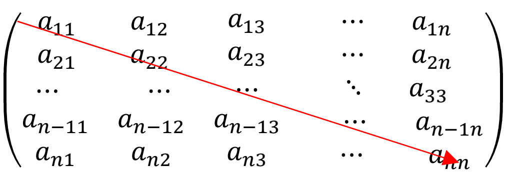
$$= \frac{1}{4}\begin{vmatrix} 4a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{vmatrix} - \frac{1}{4}\begin{vmatrix} 4a_{21} & 0 & 0 & 0 \\ 0 & a_{12} & a_{13} & a_{14} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{vmatrix} + \frac{1}{4}\begin{vmatrix} 4a_{31} & 0 & 0 & 0 \\ 0 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{42} & a_{43} & a_{44} \end{vmatrix} - \frac{1}{4}\begin{vmatrix} 4a_{41} & 0 & 0 & 0 \\ 0 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{41}\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix}$$

Using same procedure as in case 2, the determinants in above equation is obtainable. By repeating this we can obtain Surrus's rule in higher dimension ($n \times n$ matrix)

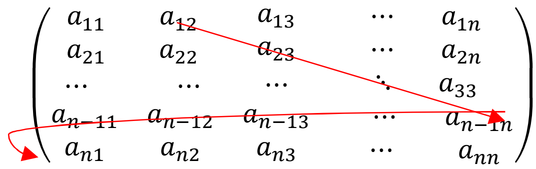
When we expand the method to n×n, the process becomes very long, though the calculation rule is simple and the author can explain how to calculate determinant by Surrus's rule using illustration.

1) Multiplicate all factors of on the line of arrow from upper left to lower right.

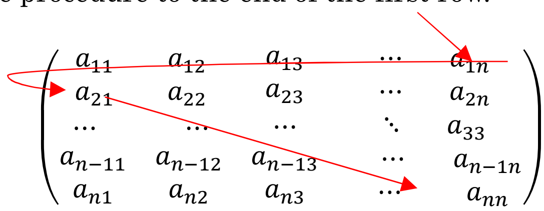



2) Multiplicate all factors on the line of arrow beginning from next factor on the first

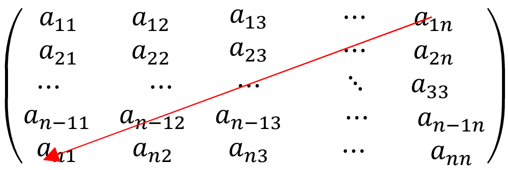
row and add the product to the value obtained by former procedure.



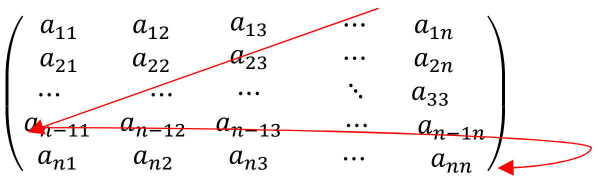

3) Repeat same procedure to the end of the first row.



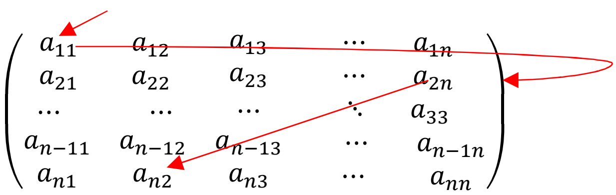

4) Multiplicate all factors on the line of the arrow stating from upper end of the matrix, and deduct the value from former value.




5) Move to the factor before former factor in fist row and deduct the vale from former value.




6) Repeat former procedure to the left end of the first row.



7) Sum up the result of 1) to 3) and subtract the sum of 4) to 6).

Following is an example in the case of $4 \times 4$.

$$\begin{vmatrix} a & b & c & d \\ g & h & i & j \\ k & l & m & n \\ s & t & u & v \end{vmatrix}$$

$$= ahmn + bins + cjkt + dglm - dils - chkv - bgnu - ajmt$$

This is Sarrus's rule. Sarrus's rule is a result of successive cofactor expansions.