V-1-6. Row reduction method

Many text books introduce "row reduction method" in early part of introduction of matrix. Row reduction method is well programed method as a calculation of determinant, and we do not need calculation of inverse matrix for solution of simultaneous equation in row reduction method. However, the method is completely the same as solution of simultaneous equation. Row reduction method is not unique for matrix calculation and is not useful for understanding function of matrix. For this reason, the author did not include row reduction method in the early part of explanation of matrix calculation. It is combination of summing and swapping of row and column. These techniques are called "basic deformation". The procedure is already explained in the paragraph of eigen vector and eigenvalue. However, row reduction method gives us important knowledge about indefinite solution and can't solve. This includes important concept in linear algebras, though the author does not enter further discussion about row reduction method and leave it at explanation of calculation method, because this is not text book of linear algebra. The author recommends readers who have interest in rank of matrix to read text books of linear algebra.

Calculation method of simultaneous equation by row reduction method.

Solution of simultaneous equation is repeats of basic deformation of rows.
Following simultaneous equation is given

$$ax + by + cz = \alpha \qquad \text{i}$$

$$kx + ly + mz = \beta \qquad \text{ii}$$

$$sx + ty + uz = \gamma \qquad \text{iii}$$

The equation can be expressed in matrix form

$$\begin{pmatrix} a & b & c \\ k & l & m \\ s & t & u \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

Make following augmented matrix

$$\begin{pmatrix} a & b & c & \vdots & \alpha \\ k & l & m & \vdots & \beta \\ s & t & u & \vdots & \gamma \end{pmatrix}$$

Make diagonal matrix left side of dotted line by swapping and summing of rows and columns.

$$\begin{pmatrix} a & b & c & \vdots & \alpha \\ k & l & m & \vdots & \beta \\ s & t & u & \vdots & \gamma \end{pmatrix}$$

Divide first row by a

$$\begin{pmatrix} 1 & \frac{b}{a} & \frac{c}{a} & \vdots & \frac{\alpha}{a} \\ k & l & m & \vdots & \beta \\ s & t & u & \vdots & \gamma \end{pmatrix}$$

second row $-\ k \times$ first row

$$\begin{pmatrix} 1 & \frac{b}{a} & \frac{c}{a} & \vdots & \frac{\alpha}{a} \\ 0 & l-\frac{b}{a}k & m-\frac{c}{a}k & \vdots & \beta-\frac{\alpha}{a}k \\ s & t & u & \vdots & \gamma \end{pmatrix}$$

Third row $-\ s \times$ first row

$$\begin{pmatrix} 1 & \frac{b}{a} & \frac{c}{a} & \vdots & \frac{\alpha}{a} \\ 0 & l-\frac{b}{a}k & m-\frac{c}{a}k & \vdots & \beta-\frac{\alpha}{a}k \\ 0 & t-\frac{b}{a}s & u-\frac{c}{a}s & \vdots & \gamma-\frac{\alpha}{a}s \end{pmatrix}$$

Divide second row by $l-\frac{b}{a}k$

$$\begin{pmatrix} 1 & \frac{b}{a} & \frac{c}{a} & \vdots & \frac{\alpha}{a} \\ 0 & 1 & \frac{ma-kc}{a}\frac{a}{la-kb} & \vdots & \frac{\beta a-\alpha k}{a}\quad\frac{a}{la-kb} \\ 0 & t-\frac{b}{a}s & u-\frac{c}{a}s & \vdots & \gamma-\frac{\alpha}{a}s \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{b}{a} & \frac{c}{a} & \vdots & \frac{\alpha}{a} \\ 0 & 1 & \frac{ma-kc}{la-kb} & \vdots & \frac{\beta a-\alpha k}{la-kb} \\ 0 & t-\frac{b}{a}s & u-\frac{c}{a}s & \vdots & \gamma-\frac{\alpha}{a}s \end{pmatrix}$$

first row $-\frac{b}{a}\times$ second row

$$\begin{pmatrix} 1 & 0 & \frac{c}{a}-\left(\frac{ma-kc}{la-kb}\right)\frac{b}{a} & \vdots & \frac{\alpha}{a}-\frac{\beta a-\alpha k}{la-kb}\frac{b}{a} \\ 0 & 1 & \frac{ma-kc}{la-kb} & \vdots & \frac{\beta a-\alpha k}{la-kb} \\ 0 & t-\frac{b}{a}s & u-\frac{c}{a}s & \vdots & \gamma-\frac{\alpha}{a}s \end{pmatrix}$$

third row $-\left(t-\frac{s}{a}b\right)\times$ second row

$$\begin{pmatrix} 1 & 0 & \frac{c}{a}-\left(\frac{ma-kc}{la-kb}\right)\frac{b}{a} & \vdots & \frac{\alpha}{a}-\frac{\beta a-\alpha k}{la-kb}\frac{b}{a} \\ 0 & 1 & \frac{ma-kc}{la-kb} & \vdots & \frac{\beta a-\alpha k}{la-kb} \\ 0 & 0 & u-\frac{c}{a}s-\left(\frac{ta-sb}{a}\right)\left(\frac{ma-kc}{la-kb}\right) & \vdots & \gamma-\frac{\alpha}{a}s-\left(\frac{ta-sb}{a}\right)\left(\frac{\beta a-\alpha k}{la-kb}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & A & \vdots & B \\ 0 & 1 & C & \vdots & D \\ 0 & 0 & E & \vdots & F \end{pmatrix}$$

$$A = \frac{c}{a}-\left(\frac{ma-kc}{la-kb}\right)\frac{b}{a} = \frac{lac-kbc-mab+kbc}{a(la-kb)} = \frac{lc-mb}{(la-kb)}$$

$$B = \frac{\alpha}{a}-\frac{\beta a-\alpha k}{la-kb}\frac{b}{a} = \frac{la\alpha-kb\alpha-\beta ab+\alpha kb}{a(la-kb)} = \frac{l\alpha-\beta b}{(la-kb)}$$

$$C = \frac{ma-kc}{la-kb}$$

$$D = \frac{\beta a-\alpha k}{la-kb}$$

$$E = u-\frac{c}{a}s-\left(\frac{ta-sb}{a}\right)\left(\frac{ma-kc}{la-kb}\right) = \frac{au-cs}{a}-\left(\frac{(ta-sb)(ma-kc)}{a(la-kb)}\right)$$

$$= \frac{(au-cs)(la-kb)-(ta-sb)(ma-kc)}{a(la-kb)}$$

$$= \frac{a^2lu-acls-abku+bcks-a^2mt+abms+ackt-bcks}{a(la-kb)}$$

$$= \frac{a(alu-cls\ -bku-amt+bms+ckt)}{a(la-kb)}$$

$$= \frac{alu+bms+ckt-cls\ -bku-amt}{la-kb}$$

$$F = \gamma-\frac{\alpha}{a}s-\left(\frac{ta-sb}{a}\right)\left(\frac{\beta a-\alpha k}{la-kb}\right)$$

$$= \frac{(a\gamma-\alpha s)(la-kb)-(ta-sb)(\beta a-\alpha k)}{a(la-kb)}$$

$$= \frac{a^2\gamma l-a\alpha ls-ab\gamma k+b\alpha ks-a^2\beta t+ab\beta s+a\alpha kt-b\alpha ks}{a(la-kb)}$$

$$= \frac{a(a\gamma l-\alpha ls-b\gamma k-a\beta t+b\beta s+\alpha kt)}{a(la-kb)}$$

$$= \frac{al\gamma + b\beta s + \alpha kt - \alpha ls - bk\gamma - a\beta t}{la - kb}$$

divide third row by E

$$\begin{pmatrix} 1 & 0 & A & \vdots & B \\ 0 & 1 & C & \vdots & D \\ 0 & 0 & 1 & \vdots & \frac{F}{E} \end{pmatrix}$$

$$\frac{F}{E} = \frac{al\gamma + b\beta s + \alpha kt - \alpha ls - bk\gamma - a\beta t}{alu + bms + ckt - cls - bku - amt}$$

second row − C × third row

$$\begin{pmatrix} 1 & 0 & A & \vdots & B \\ 0 & 1 & 0 & \vdots & D - \frac{F}{E}C \\ 0 & 0 & 1 & \vdots & \frac{F}{E} \end{pmatrix}$$

$$D - \frac{F}{E}C = \frac{\beta a - \alpha k}{la - kb} - \frac{al\gamma + b\beta s + \alpha kt - \alpha ls - bk\gamma - a\beta t}{alu + bms + ckt - cls - bku - amt} \times \frac{ma - kc}{la - kb}$$

$$= \frac{(\beta a - \alpha k)(alu + bms + ckt - cls - bku - amt) - (al\gamma + b\beta s + \alpha kt - \alpha ls - bk\gamma - a\beta t)(ma - kc)}{(la - kb)(alu + bms + ckt - cls - bku - amt)}$$

Numerator

$$\beta aalu + \beta abms + \beta ackt - \beta acls - \beta abku - \beta aamt - \alpha kalu - \alpha kbms - \alpha kckt + \alpha kcls + \alpha kbku + \alpha kamt$$

$$-maal\gamma - mab\beta s - ma\alpha kt + ma\alpha ls + mabk\gamma + maa\beta t + kcal\gamma + kcb\beta s + kc\alpha kt - kc\alpha ls - kcbk\gamma - kca\beta t$$

$$= \beta aalu - \beta acls - \beta abku - \alpha kalu - \alpha kbms + \alpha kbku - maal\gamma + ma\alpha ls + mabk\gamma + kcal\gamma + kcb\beta s - kcbk\gamma$$

$$= ala\beta u + al\alpha ms + alkc\gamma - alc\beta s - al\alpha ku - alam\gamma - kba\beta u - kb\alpha ms - kbck\gamma + kbc\beta s + kb\alpha ku + kbam\gamma$$

$$= (la - kb)(a\beta u + \alpha ms + kc\gamma - c\beta s - \alpha ku - am\gamma)$$

$$D - \frac{F}{E}C = \frac{(la - kb)(a\beta u + \alpha ms + kc\gamma - c\beta s - \alpha ku - am\gamma)}{(la - kb)(alu + bms + ckt - cls - bku - amt)} = \frac{a\beta u + \alpha ms + kc\gamma - c\beta s - \alpha ku - am\gamma}{alu + bms + ckt - cls - bku - amt}$$

first row − A × third row

$$\begin{pmatrix} 1 & 0 & 0 & \vdots & B - \frac{F}{E}A \\ 0 & 1 & 0 & \vdots & D - \frac{F}{E}C \\ 0 & 0 & 1 & \vdots & \frac{F}{E} \end{pmatrix}$$

$$B - \frac{F}{E}A = \frac{l\alpha - \beta b}{(la - kb)} - \frac{al\gamma + b\beta s + \alpha kt - \alpha ls - bk\gamma - a\beta t}{alu + bms + ckt - cls - bku - amt} \times \frac{lc - mb}{(la - kb)}$$

$$= \frac{(l\alpha - \beta b)(alu + bms + ckt - cls - bku - amt) - (al\gamma + b\beta s + \alpha kt - \alpha ls - bk\gamma - a\beta t)(lc - mb)}{(la - kb)(alu + bms + ckt - cls - bku - amt)}$$

Numerator

$$\alpha al^2u + ablms + acklt - acl^2s - \alpha blku - \alpha almt - \beta ablu - \beta b^2ms - \beta bckt + \beta bcls + \beta b^2ku + \beta abmt$$

$$-\gamma acl^2 - \beta bcls - acklt + acl^2s + \gamma bclk + \beta aclt + \gamma ablm + \beta b^2ms + \alpha bkmt - ablms - mb^2k\gamma - \beta abmt$$

$$= \alpha al^2u - \alpha blku - \alpha almt - \beta ablu - \beta bckt + \beta b^2ku - \gamma acl^2 + \gamma bclk + \beta aclt + \gamma ablm + \alpha bkmt - mb^2k\gamma$$

$$= la\alpha lu + labm\gamma + lac\beta t - lacl\gamma - lab\beta u - la\alpha mt - kb\alpha lu - kbc\beta t - kbbm\gamma + kbcl\gamma + kbb\beta u + kb\alpha mt$$

$$= (la - kb)(\alpha lu + bm\gamma + c\beta t - cl\gamma - b\beta u - \alpha mt)$$

$$B - \frac{F}{E}A = \frac{(la - kb)(\alpha lu + bm\gamma + c\beta t - cl\gamma - b\beta u - \alpha mt)}{(la - kb)(alu + bms + ckt - cls - bku - amt)} = \frac{\alpha lu + bm\gamma + c\beta t - cl\gamma - b\beta u - \alpha mt}{alu + bms + ckt - cls - bku - amt}$$

$$\begin{pmatrix} 1 & 0 & 0 & \vdots & \dfrac{\alpha lu + bm\gamma + c\beta t - cl\gamma - b\beta u - \alpha mt}{alu + bms + ckt - cls - bku - amt} \\ 0 & 1 & 0 & \vdots & \dfrac{a\beta u + \alpha ms + kc\gamma - c\beta s - \alpha ku - am\gamma}{alu + bms + ckt - cls - bku - amt} \\ 0 & 0 & 1 & \vdots & \dfrac{al\gamma + b\beta s + \alpha kt - \alpha ls - bk\gamma - a\beta t}{alu + bms + ckt - cls - bku - amt} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \dfrac{\alpha lu + bm\gamma + c\beta t - cl\gamma - b\beta u - \alpha mt}{alu + bms + ckt - cls - bku - amt} \\ \dfrac{a\beta u + \alpha ms + kc\gamma - c\beta s - \alpha ku - am\gamma}{alu + bms + ckt - cls - bku - amt} \\ \dfrac{al\gamma + b\beta s + \alpha kt - \alpha ls - bk\gamma - a\beta t}{alu + bms + ckt - cls - bku - amt} \end{pmatrix}$$

$$x = \frac{\alpha lu + bm\gamma + c\beta t - cl\gamma - b\beta u - \alpha mt}{alu + bms + ckt - cls - bku - amt} = \frac{\begin{vmatrix} \alpha & b & c \\ \beta & l & m \\ \gamma & t & u \end{vmatrix}}{\begin{vmatrix} a & b & c \\ k & l & m \\ s & t & u \end{vmatrix}}$$

$$y = \frac{a\beta u + \alpha ms + kc\gamma - c\beta s - \alpha ku - am\gamma}{alu + bms + ckt - cls - bku - amt} = \frac{\begin{vmatrix} a & \alpha & c \\ k & \beta & m \\ s & \gamma & u \end{vmatrix}}{\begin{vmatrix} a & b & c \\ k & l & m \\ s & t & u \end{vmatrix}}$$

$$z = \frac{al\gamma + b\beta s + \alpha kt - \alpha ls - bk\gamma - a\beta t}{alu + bms + ckt - cls - bku - amt} = \frac{\begin{vmatrix} a & b & \alpha \\ k & l & \beta \\ s & t & \gamma \end{vmatrix}}{\begin{vmatrix} a & b & c \\ k & l & m \\ s & t & u \end{vmatrix}}$$

We could reach the solution of simultaneous equation by row reduction method. Right side terms are expression of solution by Cramer's rule. It will be explained later paragraph. Here please confirm similarity of $\alpha lu + bm\gamma + c\beta t - cl\gamma - b\beta u - \alpha mt$ and $\begin{vmatrix} \alpha & b & c \\ \beta & l & m \\ \gamma & t & u \end{vmatrix}$.

The process looks complicate and difficult when we read the explanation, though it is not complicated and difficult in practice. As an example, cipher following simultaneous equation

$$2x + 4y + 3z = 1$$

$$x - 5y - 4z = 3$$

$$2x - y - z = 3$$

Swapping of row

$$x - 5y - 4z = 3$$

$$2x + 4y + 3z = 1$$

$$2x - y - z = 3$$

$$\begin{pmatrix} 1 & -5 & -4 \\ 2 & 4 & 3 \\ 2 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -5 & -4 & \vdots & 3 \\ 2 & 4 & 3 & \vdots & 1 \\ 2 & -1 & -1 & \vdots & 3 \end{pmatrix}$$

Second row $-$ 2 $\times$ first row, third row $-$ 2 $\times$ first row

$$\begin{pmatrix} 1 & -5 & -4 & \vdots & 3 \\ 0 & 14 & 11 & \vdots & -5 \\ 0 & 9 & 7 & \vdots & -3 \end{pmatrix}$$

Second row/14

$$\begin{pmatrix} 1 & -5 & -4 & \vdots & 3 \\ 0 & 1 & \frac{11}{14} & \vdots & \frac{-5}{14} \\ 0 & 9 & 7 & \vdots & -3 \end{pmatrix}$$

first row $-$ (-5) $\times$ second row , third row $-$ 9 $\times$ second row

$$\begin{pmatrix} 1 & 0 & -4 + \frac{55}{14} & \vdots & 3 + \frac{-25}{14} \\ 0 & 1 & \frac{11}{14} & \vdots & \frac{-5}{14} \\ 0 & 0 & 7 - \frac{99}{14} & \vdots & -3 + \frac{45}{14} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -\frac{1}{14} & \vdots & \frac{17}{14} \\ 0 & 1 & \frac{11}{14} & \vdots & \frac{-5}{14} \\ 0 & 0 & -\frac{1}{14} & \vdots & \frac{3}{14} \end{pmatrix}$$

Second row/$\left(-\frac{1}{14}\right)$

$$\begin{pmatrix} 1 & 0 & -\frac{1}{14} & \vdots & \frac{17}{14} \\ 0 & 1 & \frac{11}{14} & \vdots & \frac{-5}{14} \\ 0 & 0 & 1 & \vdots & -3 \end{pmatrix}$$

$$\text{first row} - \left(-\frac{1}{14}\right) \times \text{third row}, \text{second row} - \frac{11}{14} \times \text{third row}$$

$$\begin{pmatrix} 1 & 0 & 0 & \vdots & \frac{14}{14} \\ 0 & 1 & 0 & \vdots & \frac{28}{14} \\ 0 & 0 & 1 & \vdots & -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 1 & \vdots & -3 \end{pmatrix}$$

$$x = 1, y = 2, z = -3$$

We can obtain solution relatively short process, though the process is similar as solution method of simultaneous equation which we learned in junior high school. However, solving simultaneous equation by matrix has other meaning. Practically, using Cramer's rule we can solve simultaneous equation more automatically. More essentially, it gives us a chance to estimate the character of data set.

When following simultaneous equation is given.

$$x - 5y - 4z = 3$$

$$2x + 4y + 3z = 1$$

$$3x - y - z = 4$$

$$\begin{pmatrix} 1 & -5 & -4 & \vdots & 3 \\ 2 & 4 & 3 & \vdots & 1 \\ 3 & -1 & -1 & \vdots & 4 \end{pmatrix}$$

$\text{Second row} - 2 \times \text{first row}, \text{Third row} - 3 \times \text{first row}$

$$\begin{pmatrix} 1 & -5 & -4 & \vdots & 3 \\ 0 & 14 & 11 & \vdots & -5 \\ 0 & 14 & 11 & \vdots & -5 \end{pmatrix}$$

$\text{Second row}/14$

$$\begin{pmatrix} 1 & -5 & -4 & \vdots & 3 \\ 0 & 1 & \frac{11}{14} & \vdots & \frac{-5}{14} \\ 0 & 14 & 11 & \vdots & -5 \end{pmatrix}$$

$\text{first row} - (-5) \times \text{second row}, \text{Third row} - 14 \times \text{second row}$

$$\begin{pmatrix} 1 & 0 & \frac{-1}{14} & \vdots & \frac{17}{14} \\ 0 & 1 & \frac{11}{14} & \vdots & \frac{-5}{14} \\ 0 & 0 & 0 & \vdots & 0 \end{pmatrix}$$

All numbers in third row are 0 and we cannot make identity matrix in left side. This is because third row is sum of first and second row. Third row is not independent. In this case the solution is indefinite solution. Indefinite solution is not can't solve.

$$\begin{pmatrix} 1 & 0 & \frac{-1}{14} \\ 0 & 1 & \frac{11}{14} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{17}{14} \\ \frac{-5}{14} \\ 0 \end{pmatrix}$$

$$x - \frac{z}{14} = \frac{17}{14}$$

$$y + \frac{11}{14} z = \frac{-5}{14}$$

This means that all numbers which satisfy upper relation can be a solution. When we select $x = 1, y = 2, z = -3,$ the set of numbers satisfy the relation and the simultaneous equation.

$$1 - 5(2) - 4(-3) = 3$$
$$2(1) + 4(2) + 3(-3) = 1$$
$$3(1) - (2) - (-3) = 4$$

When we select $x = \frac{9}{7}, y = -\frac{8}{7}, z = 1,$ the set of numbers satisfy the relation and we put them in simultaneous equation.

$$\frac{9}{7} - 5\left(-\frac{8}{7}\right) - 4(1) = 3$$

$$2\left(\frac{9}{7}\right) + 4\left(-\frac{8}{7}\right) + 3(1) = 1$$

$$3\left(\frac{9}{7}\right) - \left(-\frac{8}{7}\right) - (1) = 4$$

There infinite combination of numbers which satisfy the simultaneous equation. So, the solution is infinite solution. This is simple example. the third row were made by simple sum of first row and second row. I the case of large matrix, the relation can be more complicate, and we cannot understand how the data set was produced. In such case, we can confirm whether the simultaneous equation has definite solution or infinite by row reduction method.

Calculation of determinant by row reduction method

$$\begin{vmatrix} a & b & c \\ k & l & m \\ s & t & u \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ 0 & l - \frac{bk}{a} & m - \frac{ck}{a} \\ 0 & t - \frac{bs}{a} & u - \frac{cs}{a} \end{vmatrix}$$

$$\begin{vmatrix} a & 0 & 0 \\ 0 & l-\frac{bk}{a} & m-\frac{ck}{a} \\ 0 & t-\frac{bs}{a} & u-\frac{cs}{a} \end{vmatrix}$$

$$a\begin{vmatrix} \frac{al-bk}{a} & \frac{am-ck}{a} \\ \frac{at-bs}{a} & \frac{au-cs}{a} \end{vmatrix}$$

$$= a\begin{vmatrix} \frac{al-bk}{a} & \frac{am-ck}{a} \\ 0 & \frac{au-cs}{a}-\frac{am-ck}{al-bk}\cdot\frac{at-bs}{a} \end{vmatrix}$$

$$=$$

$$= a\begin{vmatrix} \frac{al-bk}{a} & 0 \\ 0 & \frac{(au-cs)(al-bk)-(am-ck)(at-bs)}{a(al-bk)} \end{vmatrix}$$

$$(al-bk)\left|\frac{a^2lu-abku-acls+bcks-a^2mt+abms+ackt-bcks}{a(al-bk)}\right|$$

$$= (al-bk)\left|\frac{a(alu-bku-cls-amt+bms+ckt)}{a(al-bk)}\right|$$

$$= (al-bk)\left|\frac{(alu+bms+ckt-cls-bku-amt)}{(al-bk)}\right|$$

$$= alu+bms+ckt-cls-bku-amt$$

Example of 4×4 matrix

$$\begin{vmatrix} 4 & 0 & 1 & 8 \\ 1 & 2 & 0 & 4 \\ 2 & 1 & 6 & 0 \\ 0 & 4 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 0 & 4 \\ 4 & 0 & 1 & 8 \\ 2 & 1 & 6 & 0 \\ 0 & 4 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 0 & 4 \\ 0 & -8 & 1 & -8 \\ 0 & -3 & 6 & -8 \\ 0 & 4 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -8 & 1 & -8 \\ 0 & -3 & 6 & -8 \\ 0 & 4 & 1 & 2 \end{vmatrix}$$

$$= -\begin{vmatrix} -8 & 1 & -8 \\ -3 & 6 & -8 \\ 4 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -8 & -8 \\ 6 & -3 & -8 \\ 1 & 4 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -8 & 0 \\ 6 & -3 & -5 \\ 1 & 4 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 6 & 45 & -5 \\ 1 & 12 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 45 & -5 \\ 0 & 12 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} 45 & -5 \\ 12 & -2 \end{vmatrix} = \begin{vmatrix} -2 & 12 \\ -5 & 45 \end{vmatrix} = -2\begin{vmatrix} 1 & -6 \\ -5 & 45 \end{vmatrix} = -2\begin{vmatrix} 1 & -6 \\ 0 & 15 \end{vmatrix} = -2\begin{vmatrix} 1 & 0 \\ 0 & 15 \end{vmatrix} = -30$$

Calculation of inverse matrix by row reduction.

In computer calculation system, inverse matrix is generally calculated by the method using cofactor matrix which will be explained in next paragraph. However, we can obtain inverse matrix by row reduction method and it is simple and easier to understand.

In case of following matrix.

$$\begin{pmatrix} 1 & -5 & -4 \\ 2 & 4 & 3 \\ 2 & -1 & -1 \end{pmatrix}$$

Make following augment matrix

$$\left(\begin{matrix} 1 & -5 & -4 \\ 2 & 4 & 3 \\ 2 & -1 & -1 \end{matrix} \;\vdots\; \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right)$$

Basic idea is as follow. Function of inverse matrix is producing identity matrix by multiplication to original matrix. So, making identity matrix from original matrix is same as to make inverse matrix from identity matrix. We will make identity matrix in left side of augmented matrix

$\text{Second row} - 2 \times \text{first row}, \text{third row} - 2 \times \text{first row}$

$$\left(\begin{matrix} 1 & -5 & -4 \\ 0 & 14 & 11 \\ 0 & 9 & 7 \end{matrix} \;\vdots\; \begin{matrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{matrix}\right)$$

$\text{Second row}/14$

$$\left(\begin{matrix} 1 & -5 & -4 \\ 0 & 1 & \frac{11}{14} \\ 0 & 9 & 7 \end{matrix} \;\vdots\; \begin{matrix} 1 & 0 & 0 \\ -\frac{1}{7} & \frac{1}{14} & 0 \\ -2 & 0 & 1 \end{matrix}\right)$$

$\text{first row} - (-5) \times \text{Second row}, \text{third row} - 9 \times \text{second row}$

$$\left(\begin{matrix} 1 & 0 & -4 + \frac{55}{14} \\ 0 & 1 & \frac{11}{14} \\ 0 & 0 & 7 - \frac{99}{14} \end{matrix} \;\vdots\; \begin{matrix} 1 - \frac{5}{7} & \frac{5}{14} & 0 \\ -\frac{1}{7} & \frac{1}{14} & 0 \\ -2 + \frac{9}{7} & -\frac{9}{14} & 1 \end{matrix}\right)$$

$$\left(\begin{matrix} 1 & 0 & -\frac{1}{14} \\ 0 & 1 & \frac{11}{14} \\ 0 & 0 & -\frac{1}{14} \end{matrix} \;\vdots\; \begin{matrix} \frac{2}{7} & \frac{5}{14} & 0 \\ -\frac{1}{7} & \frac{1}{14} & 0 \\ -\frac{5}{7} & -\frac{9}{14} & 1 \end{matrix}\right)$$

$\text{third row}/\left(-\frac{1}{14}\right)$

$$\left(\begin{array}{ccc:ccc} 1 & 0 & -\frac{1}{14} & \frac{2}{7} & \frac{5}{14} & 0 \\ 0 & 1 & \frac{11}{14} & -\frac{1}{7} & \frac{1}{14} & 0 \\ 0 & 0 & 1 & 10 & 9 & -14 \end{array}\right)$$

first row $-\left(-\frac{1}{14}\right)\times$ third row, second row $-\ \frac{11}{14}\times$ third row

$$\left(\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{2}{7}+\frac{5}{7} & \frac{5}{14}+\frac{9}{14} & -1 \\ 0 & 1 & \frac{11}{14} & -\frac{1}{7}-\frac{110}{14} & \frac{1}{14}-\frac{99}{14} & 11 \\ 0 & 0 & 1 & 10 & 9 & -14 \end{array}\right)$$

$$\left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -\frac{112}{14} & -\frac{98}{14} & 11 \\ 0 & 0 & 1 & 10 & 9 & -14 \end{array}\right)$$

$$\left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -\frac{112}{14} & -\frac{98}{14} & 11 \\ 0 & 0 & 1 & 10 & 9 & -14 \end{array}\right)$$

$$\left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -8 & -7 & 11 \\ 0 & 0 & 1 & 10 & 9 & -14 \end{array}\right)$$

Confirmation

$$\begin{pmatrix} 1 & -5 & -4 \\ 2 & 4 & 3 \\ 2 & -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ -8 & -7 & 11 \\ 10 & 9 & -14 \end{pmatrix}$$

$$=\begin{pmatrix} 1\times 1+(-5)\times(-8)+(-4)\times 10 & 1\times 1+(-5)\times(-7)+(-4)\times 9 & 1\times(-1)+(-5)\times 11+(-4)\times(-14) \\ 2\times 1+4\times(-8)+3\times 10 & 2\times 1+4\times(-7)+3\times 9 & 2\times(-1)+4\times 11+3\times(-14) \\ 2\times 1+(-1)\times(-8)+(-1)\times 10 & 2\times 1+(-1)\times(-7)+(-1)\times 9 & 2\times(-1)+(-1)\times 11+(-1)\times(-14) \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$