V-1-8. Cofactor expansion and inverse matrix

The author already explained how to use inverse matrix and already showed several examples of making determinant and calculation of inverse matrix by row reduction method. What the author should do next is introduction of general method to make invers matrix from original matrix.　It can be summarized relatively short sentences as follow.

1. Cofactor is determinant of matrix obtained by removal of the row and column of the original factor from original matrix.
2. Cofactor matrix is a matrix of which factors are cofactors of original matrix with sign.
3. Inverse matrix is obtained by dividing transpose matrix of cofactor matrix by determinant of original matrix.

Simple but it is a jargon. The author explains the jargon using figure.

When factor is a_{ij}, we remove row i and column j from original matrix. Remove blue part from the matrix.

$$\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \cdots & \vdots \\ a_{i1} & \cdots & a_{ij} & \ddots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix}$$

Fig. 52. Explanation of cofactor (remove blue part)

Obtainable matrix is as follow

$$\begin{pmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1\,1} & \cdots & a_{i-1j-1} & a_{i-1j+1} & \cdots & a_{i-1\,n} \\ a_{i+1\,1} & \cdots & a_{i+1j-1} & a_{i+1j+1} & \cdots & a_{i+1\,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n\,j-1} & a_{n\,j+1} & \cdots & a_{n\,n} \end{pmatrix}$$

The determinant of above matrix

$$\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1\,1} & \cdots & a_{i-1j-1} & a_{i-1j+1} & \cdots & a_{i-1\,n} \\ a_{i+1\,1} & \cdots & a_{i+1j-1} & a_{i+1j+1} & \cdots & a_{i+1\,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n\,j-1} & a_{n\,j+1} & \cdots & a_{n\,n} \end{vmatrix}$$

2. We should add sign to the determinant. When subscript of original factor $(i+j)$ is even number the sign is +, and when the subscript is odd number, the sign is −. This is cofactor of a_{ij}. Here, the author introduces a unique notation rule which is applicable only in this text book for simplification and distinguishability. We express the cofactor matrix of a_{ij} as $\boldsymbol{A}^{ij}$ and cofactor as a^{ij} using red color and superscript.

$$\boldsymbol{A}^{ij} = (-1)^{i+j}\begin{pmatrix} a_{11} & \cdots & a_{1\,j-1} & a_{1\,j+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1\,1} & \cdots & a_{i-1\,j-1} & a_{i-1\,j+1} & \cdots & a_{i-1\,n} \\ a_{i+1\,1} & \cdots & a_{i+1\,j-1} & a_{i+1\,j+1} & \cdots & a_{i+1\,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n\,j-1} & a_{n\,j+1} & \cdots & a_{n\,n} \end{pmatrix}$$

$$a^{ij} = (-1)^{i+j}\begin{vmatrix} a_{11} & \cdots & a_{1\,j-1} & a_{1\,j+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1\,1} & \cdots & a_{i-1\,j-1} & a_{i-1\,j+1} & \cdots & a_{i-1\,n} \\ a_{i+1\,1} & \cdots & a_{i+1\,j-1} & a_{i+1\,j+1} & \cdots & a_{i+1\,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n\,j-1} & a_{n\,j+1} & \cdots & a_{n\,n} \end{vmatrix}$$

Then we make matrix of cofactor with sing.
Sing of each position becomes as follow.

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

This is like design of chessboard, national flag of Croatia, "Ichimatsu moyou" (a pattern of Kimono: Ichimatsu is Kabuki player in Edo period)

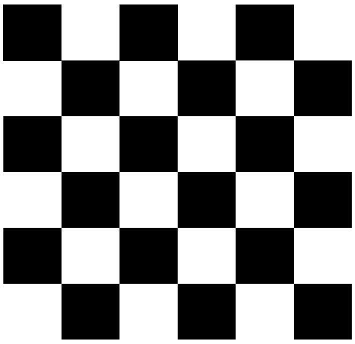

Fig. 53. Example of "Ichimatsu moyou"

In the case original matrix is $\boldsymbol{A} = \begin{pmatrix} a & b & c \\ k & l & m \\ s & t & u \end{pmatrix}$, and matrix of cofactor (C_{ij}) as **B**,

$$\boldsymbol{B}=\begin{pmatrix} \begin{vmatrix} l & m \\ t & u \end{vmatrix} & -\begin{vmatrix} k & m \\ s & u \end{vmatrix} & \begin{vmatrix} k & l \\ s & t \end{vmatrix} \\ -\begin{vmatrix} b & c \\ t & u \end{vmatrix} & \begin{vmatrix} a & c \\ s & u \end{vmatrix} & -\begin{vmatrix} a & b \\ s & t \end{vmatrix} \\ \begin{vmatrix} b & c \\ l & m \end{vmatrix} & -\begin{vmatrix} a & c \\ k & m \end{vmatrix} & \begin{vmatrix} a & b \\ k & l \end{vmatrix} \end{pmatrix}$$

Transpose matrix is matrix obtained by changing row and column of original matrix. There several notations of transpose matrix such as B^T, B^t and B'. We use B^T in this text book.

$$B^T=\begin{pmatrix} a^{11} & \cdots & a^{1j} & \cdots & a^{1n} \\ \vdots & & & & \vdots \\ a^{i1} & & a^{ij} & & a^{in} \\ \vdots & & & & \vdots \\ a^{n1} & \cdots & a^{nj} & \cdots & a^{nn} \end{pmatrix}^T=\begin{pmatrix} a^{11} & \cdots & a^{i1} & \cdots & a^{n1} \\ \vdots & & & & \vdots \\ a^{1j} & & a^{ij} & & a^{nj} \\ \vdots & & & & \vdots \\ a^{1n} & \cdots & a^{in} & \cdots & a^{nn} \end{pmatrix}$$

This is cofactor matrix of matrix $\boldsymbol{A}$, and it is expressed as $\widetilde{\boldsymbol{A}}$ in this text book.

$$\widetilde{\boldsymbol{A}}=\begin{pmatrix} a^{11} & \cdots & a^{1j} & \cdots & a^{1n} \\ \vdots & & & & \vdots \\ a^{i1} & & a^{ij} & & a^{in} \\ \vdots & & & & \vdots \\ a^{n1} & \cdots & a^{nj} & \cdots & a^{nn} \end{pmatrix}^T=\begin{pmatrix} a^{11} & \cdots & a^{i1} & \cdots & a^{n1} \\ \vdots & & & & \vdots \\ a^{1j} & & a^{ij} & & a^{nj} \\ \vdots & & & & \vdots \\ a^{1n} & \cdots & a^{in} & \cdots & a^{nn} \end{pmatrix}$$

When

$$\boldsymbol{A}=\begin{pmatrix} a & b & c \\ k & l & m \\ s & t & u \end{pmatrix}$$

then

$$\widetilde{\boldsymbol{A}}=\begin{pmatrix} \begin{vmatrix} l & m \\ t & u \end{vmatrix} & -\begin{vmatrix} b & c \\ t & u \end{vmatrix} & \begin{vmatrix} b & c \\ l & m \end{vmatrix} \\ -\begin{vmatrix} k & m \\ s & u \end{vmatrix} & \begin{vmatrix} a & c \\ s & u \end{vmatrix} & -\begin{vmatrix} a & c \\ k & m \end{vmatrix} \\ \begin{vmatrix} k & l \\ s & t \end{vmatrix} & -\begin{vmatrix} a & b \\ s & t \end{vmatrix} & \begin{vmatrix} a & b \\ k & l \end{vmatrix} \end{pmatrix}$$

Formula 52

3. Divide cofactor matrix ($\widetilde{\boldsymbol{A}}$) by determinant of original matrix ($|\boldsymbol{A}|$), inverse matrix is obtained.

$$\boldsymbol{A}^{-1}=\frac{\widetilde{\boldsymbol{A}}}{|\boldsymbol{A}|}$$

Formula 53

As an example of 3×3,

$$\boldsymbol{A}=\begin{pmatrix} a & b & c \\ k & l & m \\ s & t & u \end{pmatrix}$$

$$A^{-1} = \frac{1}{\begin{vmatrix} a & b & c \\ k & l & m \\ s & t & u \end{vmatrix}} \begin{pmatrix} \begin{vmatrix} l & m \\ t & u \end{vmatrix} & -\begin{vmatrix} b & c \\ t & u \end{vmatrix} & \begin{vmatrix} b & c \\ l & m \end{vmatrix} \\ -\begin{vmatrix} k & m \\ s & u \end{vmatrix} & \begin{vmatrix} a & c \\ s & u \end{vmatrix} & -\begin{vmatrix} a & c \\ k & m \end{vmatrix} \\ \begin{vmatrix} k & l \\ s & t \end{vmatrix} & -\begin{vmatrix} a & b \\ s & t \end{vmatrix} & \begin{vmatrix} a & b \\ k & l \end{vmatrix} \end{pmatrix}$$

Formula 54

Calculations of determinant and inverse matrix are troublesome works. The calculation by hand is nearly impossible. The author recommends use of computer soft wares such as Microsoft Excel. Honestly, the author seldom calculates determinant and inverse matrix by himself. However, concept of cofactor matrix is important for deeper understanding of linear algebra spaces.

Multiply $\boldsymbol{A}$ to both side of equation 53.

$$\boldsymbol{A}^{-1} = \frac{\widetilde{\boldsymbol{A}}}{|\boldsymbol{A}|}$$

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}\frac{\widetilde{\boldsymbol{A}}}{|\boldsymbol{A}|}$$

Left side is unit matrix from the definition of inverse matrix.

$$\boldsymbol{I} = \frac{\boldsymbol{A}\widetilde{\boldsymbol{A}}}{|\boldsymbol{A}|}$$

$$\boldsymbol{A}\widetilde{\boldsymbol{A}} = |\boldsymbol{A}|\boldsymbol{I}$$

This transformation means that product of matrix and its cofactor matrix is product of determinant and unit matrix. From this, we discuss the relation of inverse matrix and cofactor matrix.

We already use concept of cofactor in V-1-5. Suruss's rule implicitly.

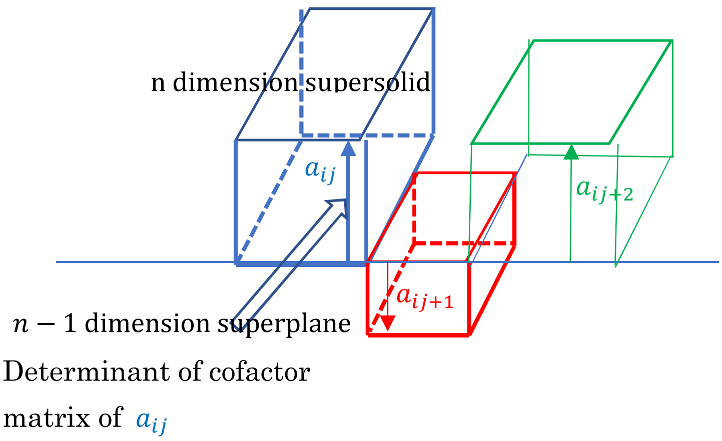

Fig. 51. Concept of Surruss's rule

Linear algebra is mathematics in multidimensional space. It is impossible illustrate multidimensional figure in plane. Thus, the author express 3-dimensional illustration in figure 51. We consider a plane and vectors orthogonal to the plane. We calculate volume of boxes in figure 51 by multiplying length of vectors to area of tetragon on the plane. We consider the magnitudes of the boxes are determinants. This is a kind of analogy. When we consider 3-dimensional space as n-dimensional space, flat plane is $(n-1)$-dimensional space and area on the plane is magnitude of the $(n-1)$-dimensional solid. Magnitude of n-dimensional solid is obtainable by multiplying length of the vector to the magnitude of $(n-1)$-dimensional solid. As in the illustration of 2-dimensional solid in figure 50, the sing of the magnitude of the solid changes of dimension. A vector can be expressed by linear combination of orthogonal vectors. When we consider a factor in matrix as length of a vector, we calculate magnitude of n-dimensional solid by multiplying the factor to the determinant of matrix which obtained by removing the row and line on which the factor is exist, because the determinant is $(n-1)$-dimensional magnitude.

The author will show the transformation of matrix depending on above idea. The first step of the transformation is the same as the transformation explained in V-1-5. "Suruss's rule.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11}+0+\cdots+0 & a_{12} & \cdots & a_{1n} \\ a_{21}+0+\cdots+0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+0+\cdots+0 & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11}+0+\cdots+0 & a_{12} & \cdots & a_{1n} \\ 0+a_{21}+\cdots+0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0+\cdots+0+a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \frac{1}{n}\begin{vmatrix} na_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \frac{1}{n}\begin{vmatrix} 0 & a_{12} & \cdots & a_{1n} \\ na_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \cdots + \frac{1}{n}\begin{vmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ na_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

We swap the row depending the rule of matrix calculation. The sing of the determinant changes by each swapping.

$$= \frac{(-1)^0}{n}\begin{vmatrix} na_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \frac{(-1)^1}{n}\begin{vmatrix} na_{21} & a_{22} & \cdots & a_{2n} \\ 0 & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \cdots + \frac{(-1)^{n-1}}{n}\begin{vmatrix} na_{n1} & a_{n2} & \cdots & a_{nn} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n-1\,2} & \cdots & a_{n-1\,n} \end{vmatrix}$$

Row reduction method: Subtract product of first column and coefficient from other column.

$$= \frac{(-1)^0}{n}\begin{vmatrix} na_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \frac{(-1)^1}{n}\begin{vmatrix} na_{21} & 0 & \cdots & 0 \\ 0 & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \cdots + \frac{(-1)^{n-1}}{n}\begin{vmatrix} na_{n1} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n-1\,2} & \cdots & a_{n-1\,n} \end{vmatrix}$$

As explained in figure 51, the determinants of upper equation can be transformed as follow.

$$= \frac{(-1)^0}{n}\begin{vmatrix} na_{11} & 0 \\ 0 & \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} \end{vmatrix} + \frac{(-1)^1}{n}\begin{vmatrix} na_{21} & 0 \\ 0 & \begin{vmatrix} a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} \end{vmatrix} + \cdots + \frac{(-1)^{n-1}}{n}\begin{vmatrix} na_{n1} & 0 \\ 0 & \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n-1\,2} & \cdots & a_{n-1\,n} \end{vmatrix} \end{vmatrix}$$

$$= (-1)^0 a_{11} \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} + (-1)^1 a_{21} \begin{vmatrix} a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} + \cdots + (-1)^{n-1} a_{n1} \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n-1\,2} & \cdots & a_{n-1\,n} \end{vmatrix}$$

$$= (-1)^0 a_{11} a^{11} + (-1)^1 a_{21} a^{21} + \cdots + (-1)^{n-1} a_{n1} a^{n1}$$

We do same procedure in the other column

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= (-1)^0 a_{11} a^{11} + (-1)^1 a_{12} a^{12} + \cdots + (-1)^{n-1} a_{12} a^{1n}$$

$$+(-1)^1 a_{21} a^{21} + (-1)^2 a_{22} a^{22} + (-1)^n a_{12} a^{2n}$$

$$\vdots$$

$$+(-1)^{n-1} a_{n1} a^{n1} + (-1)^n a_{n1} a^{n2} + \cdots (-1)^{2n-2} a_{12} a^{nn}$$

This is determinant of following matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} (-1)^{1+1-2} a^{11} & (-1)^{1+2-2} a^{21} & \cdots & (-1)^{1+2-2} a^{n1} \\ (-1)^{2+1-2} a^{12} & (-1)^{2+2-2} a^{22} & \cdots & (-1)^{2+n-2} a^{n2} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+1-2} a^{1n} & (-1)^{n+2-2} a^{2n} & \cdots & (-1)^{n+n-2} a^{nn} \end{pmatrix}$$

The second matrix is cofactor matrix.

$$\tilde{\boldsymbol{A}} = \begin{pmatrix} (-1)^{1+1-2} a^{11} & (-1)^{1+2-2} a^{21} & \cdots & (-1)^{1+2-2} a^{n1} \\ (-1)^{2+1-2} a^{12} & (-1)^{2+2-2} a^{22} & \cdots & (-1)^{2+n-2} a^{n2} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+1-2} a^{1n} & (-1)^{n+2-2} a^{2n} & \cdots & (-1)^{n+n-2} a^{nn} \end{pmatrix}$$

$$|\boldsymbol{A}\tilde{\boldsymbol{A}}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = |\boldsymbol{A}|$$

Dividing both sides by $|\boldsymbol{A}|$.

$$\left|\boldsymbol{A}\frac{\tilde{\boldsymbol{A}}}{|\boldsymbol{A}|}\right| = 1$$

We make original matrix from upper determinant. For this, we change right side with unit matrix.

$$\boldsymbol{A}\frac{\tilde{\boldsymbol{A}}}{|\boldsymbol{A}|} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Obviously $\frac{\tilde{\boldsymbol{A}}}{|\boldsymbol{A}|}$ is inverse matrix of $\boldsymbol{A}$.

$$\boldsymbol{A}^{-1} = \frac{1}{|\boldsymbol{A}|} \begin{pmatrix} a^{11} & a^{21} & \cdots & a^{n1} \\ a^{12} & a^{22} & \cdots & a^{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a^{1n} & a^{2n} & \cdots & a^{nn} \end{pmatrix}$$

以上が、余因子行列を行列式で割ったのものが、逆行列であることの証明です。