V-1-9. Cramer' s rule

We already know Cramer's rule through learning calculation techniques of matrix. An example of Cramer's rule is as follow
In the case of following 3 unknowns,

$$ax + by + cz = \alpha \qquad \text{i}$$

$$kx + ly + mz = \beta \qquad \text{ii}$$

$$sx + ty + uz = \gamma \qquad \text{iii}$$

$$\begin{pmatrix} a & b & c \\ k & l & m \\ s & t & u \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

,

$$x = \frac{\begin{vmatrix} \alpha & b & c \\ \beta & l & m \\ \gamma & t & u \end{vmatrix}}{\begin{vmatrix} a & b & c \\ k & l & m \\ s & t & u \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a & \alpha & c \\ k & \beta & m \\ s & \gamma & u \end{vmatrix}}{\begin{vmatrix} a & b & c \\ k & l & m \\ s & t & u \end{vmatrix}}, \qquad z = \frac{\begin{vmatrix} a & b & \alpha \\ k & l & \beta \\ s & t & \gamma \end{vmatrix}}{\begin{vmatrix} a & b & c \\ k & l & m \\ s & t & u \end{vmatrix}}$$

We can get solution simply and automatically. This is Cramer's rule. This rule can be applied to n-unknown simultaneous equation. Cramer's rule is one of the most important rules in matrix calculation. General expression of Cramer's rule is as follow.
When

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad \boldsymbol{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}. \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

and

$$\boldsymbol{AX} = \boldsymbol{b}$$

,

$$x_i = \frac{|\boldsymbol{A_i}|}{|\boldsymbol{A}|}$$

Here, definition of $\boldsymbol{A_i}$ is as follow.

$$\boldsymbol{A_i} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \cdots & a_{1,n} \\ a_{2,1} & \cdots & a_{2,i-1} & b_2 & a_{2,i+1} & \cdots & a_{2,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,i-1} & b_n & a_{n,i+1} & \cdots & a_{n,n} \end{pmatrix}$$

Formula 56

The procedure is replacement of column i with $\boldsymbol{b}$ in numerator determinant of x_i

Using our unique notation rule in this text book.

$$\boldsymbol{A^{ij}} = (-1)^{ij}\begin{pmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1} & \cdots & a_{i-1j-1} & a_{i-1j+1} & \vdots & a_{i-1n} \\ a_{i+11} & \cdots & a_{i+1j-1} & a_{i+1j+1} & \cdots & a_{i+1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj+1} & \cdots & a_{nn} \end{pmatrix}$$

$$a^{ij} = (-1)^{ij}\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{i-11} & \cdots & a_{i-1j-1} & a_{i-1j+1} & \vdots & a_{i-1n} \\ a_{i+11} & \cdots & a_{i+1j-1} & a_{i+1j+1} & \cdots & a_{i+1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}$$

$$\boldsymbol{A^{-1}} = \frac{1}{|\boldsymbol{A}|}\begin{pmatrix} a^{11} & a^{21} & \cdots & a^{n1} \\ a^{12} & a^{22} & \cdots & a^{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a^{1n} & a^{2n} & \cdots & a^{nn} \end{pmatrix}$$

$$\boldsymbol{AX} = \boldsymbol{b}$$

$$\boldsymbol{A}^{-1}\boldsymbol{AX} = \boldsymbol{A}^{-1}\boldsymbol{b}$$

$$\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{b}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{|A|}\begin{pmatrix} a^{11} & a^{21} & \cdots & a^{n1} \\ a^{12} & a^{22} & \cdots & a^{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a^{1n} & a^{2n} & \cdots & a^{nn} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{|A|}\begin{pmatrix} b_1a^{11} + b_2a^{21} + \cdots + b_na^{n1} \\ b_1a^{12} + b_2a^{22} + \cdots + b_na^{22} \\ \vdots \\ b_1a^{1n} + b_2a^{2n} + \cdots + b_na^{nn} \end{pmatrix}$$

For x_i

$$x_i = \frac{1}{|A|}\left(b_1a^{1i} + b_2a^{2i} + \cdots + b_na^{ni}\right)$$

On the other hand, cofactor $b_1a^{1i} + b_2a^{2i} + \cdots + b_na^{ni}$ is cofactor expansion of following matrix at column *i*.

$$\begin{pmatrix} a_{11} & \cdots & a_{1i-1} & b_1 & a_{1i+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2i-1} & b_2 & a_{2i+1} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{ni-1} & b_n & a_{ni+1} & \cdots & a_{nn} \end{pmatrix}$$

Example

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

Cofactor expansion at column 3.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

$$= a_{13}(-1)^{1+3}\begin{pmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{pmatrix} + a_{23}(-1)^{2+3}\begin{pmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{pmatrix} + a_{33}(-1)^{3+3}\begin{pmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{pmatrix}$$

$$+ a_{43}(-1)^{4+3}\begin{pmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{pmatrix}$$

$$= a_{13}\boldsymbol{A}^{\mathbf{13}} + a_{23}\boldsymbol{A}^{\mathbf{23}} + a_{33}\boldsymbol{A}^{\mathbf{33}} + a_{43}\boldsymbol{A}^{\mathbf{43}}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = a_{13}a^{13} + a_{23}a^{23} + a_{33}a^{33} + a_{43}a^{43}$$

Replacement of colmn 3 with $\boldsymbol{b}$

$$\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & b_1 & a_{14} \\ a_{21} & a_{22} & b_2 & a_{24} \\ a_{31} & a_{32} & b_3 & a_{34} \\ a_{41} & a_{42} & b_4 & a_{44} \end{pmatrix} = b_1\boldsymbol{A}^{\mathbf{13}} + b_2\boldsymbol{A}^{\mathbf{23}} + b_3\boldsymbol{A}^{\mathbf{33}} + b_4\boldsymbol{A}^{\mathbf{43}}$$

$$\begin{vmatrix} a_{11} & a_{12} & b_1 & a_{14} \\ a_{21} & a_{22} & b_2 & a_{24} \\ a_{31} & a_{32} & b_3 & a_{34} \\ a_{41} & a_{42} & b_4 & a_{44} \end{vmatrix} = b_1a^{13} + b_2a^{23} + b_3a^{33} + b_4a^{43}$$

$$x_i = \frac{1}{|A|}\left(b_1a^{1i} + b_2a^{2i} + \cdots + b_na^{ni}\right) = \frac{\begin{vmatrix} a_{11} & \cdots & a_{1i-1} & b_1 & a_{1i+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2i-1} & b_2 & a_{2i+1} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{ni-} & b_n & a_{ni+} & \cdots & a_{nn} \end{vmatrix}}{|A|}$$