V-1-6. Eigenvector and eigenvalue

Eigenvector and eigenvalue are important concept in matrix calculation, particularly application of matrix calculation in multi-valuable analysis. The reader who want to obtain skill of multi-valuable analysis need to learn eigenvector and eigenvalue. The readers are already familiar with matrix calculation. Learning of eigenvector and eigenvalue is rather easy.

We go back to story of the navigation "go straight 100m and turn to the left, then go to 200m. In this example, the road is not cross orthogonally, and two roads cross having $\frac{\pi}{3}$ angle.
We multiply following matrix from left to the vectors in the map in brain to match point of real map.

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$$

This transformation can be expressed following illustration.

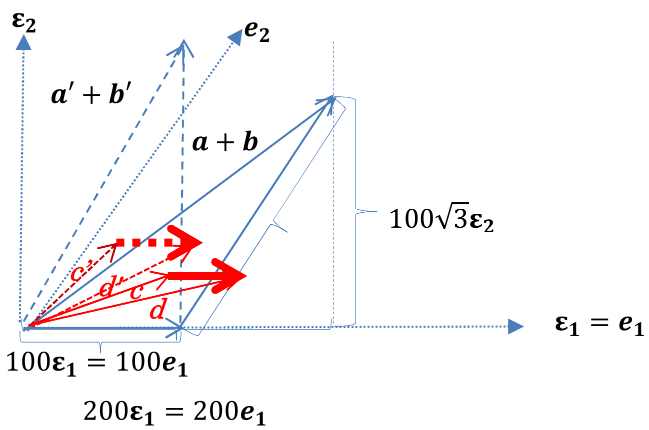

Fig. 55. Example of eigen vector.

The triangle formed by vectors are deformed transformation, though the direction of dotted red arrow is not changed by multiplication of the matrix.
Confirmation
Dotted red arrow is $\boldsymbol{d}' - \boldsymbol{c}'$

$$\boldsymbol{d}' = \begin{pmatrix} 100 \\ 50 \end{pmatrix}, \ \boldsymbol{c}' = \begin{pmatrix} 50 \\ 50 \end{pmatrix}$$

$$\boldsymbol{d}' - \boldsymbol{c}' = \begin{pmatrix} 100 \\ 50 \end{pmatrix} - \begin{pmatrix} 50 \\ 50 \end{pmatrix} = \begin{pmatrix} 50 \\ 0 \end{pmatrix}$$

$$\boldsymbol{d} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix} \boldsymbol{d'} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 100 \\ 50 \end{pmatrix} = \begin{pmatrix} 125 \\ 25\sqrt{3} \end{pmatrix}$$

$$\boldsymbol{c} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix} \boldsymbol{c'} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 50 \\ 50 \end{pmatrix} = \begin{pmatrix} 75 \\ 25\sqrt{3} \end{pmatrix}$$

$$\boldsymbol{d} - \boldsymbol{c} = \begin{pmatrix} 125 \\ 25\sqrt{3} \end{pmatrix} - \begin{pmatrix} 75 \\ 25\sqrt{3} \end{pmatrix} = \begin{pmatrix} 50 \\ 0 \end{pmatrix}$$

In this case, both direction and length were not changed. This a special case. The definition of eigenvector is no change in direction. The ratio of changes in length is eigenvalue. In this case eigenvalue is 1. Originally, eigen is German meaning characteristic or unique

Matrix formula of change in length and no change in direction is as follow

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix}$$

λ :eigen value

In the form of simultaneous equation

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = \lambda x_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = \lambda x_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = \lambda x_2$$

Transposition right term to left

$$\begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

This means determinant of left side should be 0.

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$

Formula 57

Formula 57 is equation to calculate eigenvalue. The name of the formula is characteristic equation. We can obtain λ by characteristic equation. From this equation, it is obvious that $n \times n$ matrix has n eigen values, though they not always real number. When multiple root exists the number of eigen value decrease by number of multiple roots. Eigenvector not always exists in the space of matrix. Sometimes it exists in new dimension. The matrix lining up eigen values on diagonal line is eigenvalue matrix and expressed by $\boldsymbol{\Lambda}$ (lambda not A).

$$\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

Eigenvectors are obtainable by solving following equation from the definition of eigen vector.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

When we implement the solution of above equation, we will face a confusion. This confusion comes from the nature of vector. Vector is not scalar. Vectors obtained by multiplication of scalar to eigenvector is eigen vectors. Solution is given as relation of components on vectors. The author exemplifies the form of the solution in the story of guidance.

$$\begin{vmatrix} 1-\lambda & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} - \lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left(\frac{\sqrt{3}}{2} - \lambda\right) = 0$$

$$\lambda = 1, \qquad \lambda = \frac{\sqrt{3}}{2}$$

This case is particularly unique. When we put $\lambda = 1$

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x_1 + \frac{1}{2} x_2 = x_1$$

$$\frac{\sqrt{3}}{2} x_2 = x_2$$

From both

$$x_2 = 0, x_1 \text{: arbitrary real number}$$

This can be expressed as follow

$$\begin{pmatrix} t \\ 0 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The other eigenvector belonging $\lambda = \frac{\sqrt{3}}{2}$

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{\sqrt{3}}{2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x_1 + \frac{1}{2} x_2 = \frac{\sqrt{3}}{2} x_2$$

$$\frac{\sqrt{3}}{2} x_2 = \frac{\sqrt{3}}{2} x_2$$

$$x_1 = \frac{\sqrt{3} - 1}{2} x_2$$

$$t \begin{pmatrix} \frac{\sqrt{3} - 1}{2} \\ 1 \end{pmatrix}$$

t: arbitrary real number

From this calculation we could confirm accuracy of our impression from figure 52 that the vector parallel to x_1 axis is not changed in direction by multiplication of the matrix. More accurately, parallel vectors with following two arrows are not changed by the multiplication of the matrix.

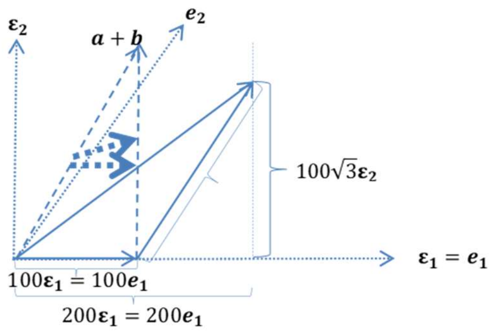

Fig. 56. Eigen vectors

Above example is too unique. More general example is as follow.

$$\boldsymbol{A} = \begin{pmatrix} 5 & -6 \\ 2 & -2 \end{pmatrix}$$

Eigenvalue

$$\begin{vmatrix} 5-\lambda & -6 \\ 2 & -2-\lambda \end{vmatrix}$$

$$= (5-\lambda)(-2-\lambda) + 12$$

$$= \lambda^2 - 3\lambda - 10 + 12$$

$$= (\lambda - 2)(\lambda - 1) = 0$$

$$\lambda = 2 \text{ or } \lambda = 1$$

$$\begin{pmatrix} 5 & -6 \\ 2 & -2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Eigenvector belonging in eigenvalue $\lambda = 2$

$$\begin{pmatrix} 5 & -6 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$5x_1 - 6x_2 = 2x_1$$

$$2x_1 - 2x_2 = 2x_2$$

$$2x_2 = x_1$$

The eigenvector belonging in $\lambda = 2$ is

$$t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Eigenvector belonging in eigenvalue $\lambda = 1$

$$\begin{pmatrix} 5 & -6 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$5x_1 - 6x_2 = x_1$$

$$2x_1 - 2x_2 = x_2$$

$$2x_1 = 3x_2$$

$$2x_1 - 3x_2 = 0$$

The eigenvector is belonging in eigenvalue $\lambda = 1$ is

$$\mathrm{t}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Conclusively, eigenvector is obtainable as solution of following equation.

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$

The equation can be generalized and simplified as follow

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0$$

$\boldsymbol{I}$: identity matrix

This is characteristic equation.

When we change λ with t the name is characteristic polynomial $|\boldsymbol{A} - t\boldsymbol{I}|$

Generally, characteristic polynomial of t is expressed as $\phi_A(t)$

$$\phi_A(t) = |\boldsymbol{A} - t\boldsymbol{I}|$$

When matrix $\boldsymbol{A}$ and matrix $\boldsymbol{B}$ are similar

$$\phi_A(t) = \phi_B(t)$$

More accurately, identity of eigenvalues is definition of similarity of matrixes.