V-2. Basic linear algebras

V-2-1. Similarity

Similarity

Definition

When matrix $\boldsymbol{C}$ and $\boldsymbol{D}$ are in following relation, the matrixes are similar.

$$\boldsymbol{C} = \boldsymbol{P}^{-1}\boldsymbol{D}\boldsymbol{P}$$

Formula 59

In more practical expression, when such $\boldsymbol{P}$ exists $\boldsymbol{C}$ and $\boldsymbol{D}$ are similar. The author explains meaning of definition exemplifying a 3×3 matrix.

$$\boldsymbol{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{P} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Calculation of determinant, transpose and inverse matrix.

$$|\boldsymbol{P}| = (2 + 2 + 1) - (1 + 4 + 1) = -1$$

$$\boldsymbol{P}^T = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\boldsymbol{P}^{-1} = \frac{1}{|P|}\begin{pmatrix} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \end{pmatrix}$$

$$= \frac{1}{-1}\begin{pmatrix} 1 & -3 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 3 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

Calculation of $\boldsymbol{C}$

$$\boldsymbol{C} = \boldsymbol{P}^{-1}\boldsymbol{D}\boldsymbol{P}$$

$$\boldsymbol{P}^{-1}\boldsymbol{D} = \begin{pmatrix} -1 & 3 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 12 & -1 \\ 3 & -4 & 0 \\ 0 & -4 & 1 \end{pmatrix}$$

$$\boldsymbol{P}^{-1}\boldsymbol{D}\boldsymbol{P} = \begin{pmatrix} -3 & 12 & -1 \\ 3 & -4 & 0 \\ 0 & -4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -3 + 12 - 1 & -6 + 12 - 1 & -3 + 12 - 2 \\ 3 - 4 + 0 & 6 - 4 + 0 & 3 - 4 + 0 \\ 0 - 4 + 1 & 0 - 4 + 1 & 0 - 4 + 2 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 5 & 7 \\ -1 & 2 & -1 \\ -3 & -3 & -2 \end{pmatrix}$$

$$\boldsymbol{C} = \begin{pmatrix} 8 & 5 & 7 \\ -1 & 2 & -1 \\ -3 & -3 & -2 \end{pmatrix}$$

Then, we can say that $\boldsymbol{C}$ and $\boldsymbol{D}$ **are similar.**

We can understand the meaning of definition, though we still do not understand how we can use the similarity practically. The most important practical meaning of similarity among matrix is similarity of eigen value. Eigen vectors and eigen values of $\boldsymbol{D}$ are trivial.

$$\lambda = 4,3,1$$

Eigenvector

$$\lambda = 4, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (0 \quad 1 \quad 0)^T$$

$$\lambda = 3, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (1 \quad 0 \quad 0)^T$$

$$\lambda = 1, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (0 \quad 0 \quad 1)^T$$

For $\boldsymbol{C}$

$$\begin{vmatrix} (8-\lambda) & 5 & 7 \\ -1 & (2-\lambda) & -1 \\ -3 & -3 & (-2-\lambda) \end{vmatrix} = 0$$

$$\lambda^3 - 8\lambda^2 + 19\lambda - 12 = 0$$

$$(\lambda - 4)(\lambda - 3)(\lambda - 1) = 0$$

$$\lambda = 4, 3, 1$$

Eigenvector

$$\begin{pmatrix} 8 & 5 & 7 \\ -1 & 2 & -1 \\ -3 & -3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

When $\lambda = 4$,

$$8x_1 + 5x_2 + 7x_3 = 4x_1$$

$$-1x_1 + 2x_2 - x_3 = 4x_2$$

$$-3x_1 - 3x_2 - 2x_3 = 4x_3$$

$$4x_1 + 5x_2 + 7x_3 = 0 \qquad \text{i}$$

$$-x_1 - 2x_2 - x_3 = 0 \qquad \text{ii}$$

$$-3x_1 - 3x_2 - 6x_3 = 0 \qquad \text{iii}$$

$\text{i} + 4 \times \text{iii}$

$$-3x_2 + 3x_3 = 0$$

$$x_2 = x_3 \qquad \text{ii'}$$

Put ii' in iii

$$-3x_1 - 9x_2 = 0$$

$$x_1 = -3x_2$$

$$\text{Eigenvalue } (\lambda = 4)\text{: t}\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$

t: arbitrary real number

When λ = 3

$$8x_1 + 5x_2 + 7x_3 = 3x_1$$

$$-1x_1 + 2x_2 - x_3 = 3x_2$$

$$-3x_1 - 3x_2 - 2x_3 = 3x_3$$

$$5x_1 + 5x_2 + 7x_3 = 0$$

$$-x_1 - x_2 - x_3 = 0$$

$$-3x_1 - 3x_2 - 5x_3 = 0$$

$$x_3 = 0$$

$$x_1 = -x_2$$

$$\text{Eigenvalue } (\lambda = 3)\text{: t}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

When λ = 1、

$$8x_1 + 5x_2 + 7x_3 = x_1$$

$$-1x_1 + 2x_2 - x_3 = x_2$$

$$-3x_1 - 3x_2 - 2x_3 = x_3$$

$$7x_1 + 5x_2 + 7x_3 = 0$$

$$-x_1 + x_2 - x_3 = 0$$

$$-3x_1 - 3x_2 - 3x_3 = 0$$

$$x_2 = 0$$

$$x_1 = -x_3$$

$$\text{Eigen vector } (\lambda = 1)\text{: t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Let us denote each eigenvector of $\boldsymbol{D}$ as follows.

$$\boldsymbol{E_{d1}} = (1 \quad 0 \quad 0)^T$$

$$\boldsymbol{E_{d2}} = (0 \quad 1 \quad 0)^T$$

$$\boldsymbol{E_{d3}} = (0 \quad 0 \quad 1)^T$$

and eigenvector of $\boldsymbol{C}$ as follows

$$\boldsymbol{E_{c1}} = (1 \quad -1 \quad 0)^T$$

$$\boldsymbol{E_{c2}} = (3 \quad -1 \quad -1)^T$$

$$\boldsymbol{E_{c3}} = (-1 \quad 0 \quad 1)^T$$

(The author puts $t = -1$ in $\boldsymbol{E_{c3}}$, and puts $t = 1$ in $\boldsymbol{E_{c1}}$ and $\boldsymbol{E_{c2}}$ without any mathematical reason. That is only for convenience in following explanation.)

Calculation of inner products among eigenvectors of $\boldsymbol{D}$

Example

$$\boldsymbol{E_{d1}} \cdot \boldsymbol{E_{d2}} = \boldsymbol{E_{d1}}^{\boldsymbol{T}} \boldsymbol{E_{d2}} = (1 \quad 0 \quad 0)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$\boldsymbol{E_{d1}} \cdot \boldsymbol{E_{d2}} = \boldsymbol{E_{d2}} \cdot \boldsymbol{E_{d3}} = \boldsymbol{E_{d3}} \cdot \boldsymbol{E_{d1}} = 0$$

$$\boldsymbol{P_1}' \cdot \boldsymbol{P_2}' \neq 0, \boldsymbol{P_2}' \cdot \boldsymbol{P_3}' \neq 0, \boldsymbol{P_3}' \cdot \boldsymbol{P_1}' \neq 0$$

Calculation of inner products among eigenvectors of $\boldsymbol{C}$

$$\boldsymbol{E_{c1}} \cdot \boldsymbol{E_{c2}} = \boldsymbol{E_{c1}}^{\boldsymbol{T}} \boldsymbol{E_{c2}} = (1 \quad -1 \quad 0)\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = 1 \times 3 \times (-1) + 0 \times (-1) = 7$$

$$\boldsymbol{E_{c2}} \cdot \boldsymbol{E_{c3}} = \boldsymbol{E_{c2}}^{\boldsymbol{T}} \boldsymbol{E_{c3}} = (3 \quad -1 \quad -1)\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 3 \times (-1) + (-1) \times 0 + (-1) \times (1) = -4$$

$$\boldsymbol{E_{c3}} \cdot \boldsymbol{E_{c1}} = \boldsymbol{E_{c3}}^{\boldsymbol{T}} \boldsymbol{E_{c1}} = (-1 \quad 0 \quad 1)\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = (-1) \times 1 + 0 \times (-1) + (1) \times 0 = -1$$

This means that eigenvectors of $\boldsymbol{D}$ are orthogonal each other, though eigenvectors of $\boldsymbol{C}$ are not orthogonal.

Conclusively, we can say that eigenvalues are similar between similar matrix, though the angles between vectors in each matrix are not always similar.