V-2-2. Diagonalization

The definition of similarity is as follow.

$$\boldsymbol{C} = \boldsymbol{P}^{-1}\boldsymbol{D}\boldsymbol{P}$$

The author exemplified the relation following matrixes

$$\boldsymbol{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{C} = \begin{pmatrix} 8 & 5 & 7 \\ -1 & 2 & -1 \\ -3 & -3 & -2 \end{pmatrix}$$

Matrix $\boldsymbol{D}$ is diagonal matrix in this exemplified case, though it is not necessarily to be diagonal matrix generally. This definition means that matrix $\boldsymbol{D}$ is similar with matrix $\boldsymbol{C}$, because of existence of $\boldsymbol{P}$. In this particular case, we can say that diagonal matrix $\boldsymbol{D}$ is similar with non-diagonal matrix $\boldsymbol{C}$.

When we multiply $\boldsymbol{P}$ from left side and $\boldsymbol{P}^{-1}$ from right side to both sides of the equation

$$\boldsymbol{P}\boldsymbol{C}\boldsymbol{P}^{-1} = \boldsymbol{P}\boldsymbol{P}^{-1}\boldsymbol{D}\boldsymbol{P}\boldsymbol{P}^{-1}$$

$$\boldsymbol{P}\boldsymbol{C}\boldsymbol{P}^{-1} = \boldsymbol{D}$$

When we consider

$$\boldsymbol{P}^{-1} = \boldsymbol{Q}$$

then

$$\boldsymbol{P} = \boldsymbol{Q}^{-1}$$

$$\mathbf{Q}^{-1}\mathbf{C}\mathbf{Q} = \boldsymbol{D}$$

There exists various similar matrix. However, we can say that we can make diagonal matrix from non-diagonal matrix, when we can find proper $\boldsymbol{Q}$ from the transformed definition. A procedure to make diagonal matrix from non-diagonal matrix by multiplying a matrix from right side and its inverse matrix from left side is named diagnalization.

Mathematical definition of diagonalization is as follow.

$$\boldsymbol{Q}^{-1}\boldsymbol{C}\boldsymbol{Q} = \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_p \end{pmatrix}$$

Formula 60

When there exists $\boldsymbol{Q}$, $\boldsymbol{C}$ can be diagonalized.

(The author wants to use n as dimension of matrix, though n is generally used as sample size. For the prevention of confusion, p is used as dimension of matrix here.).

We consider matrix $\boldsymbol{Q}$ is a row of vectors $\boldsymbol{Q_i}$.

$$\boldsymbol{Q} = (\boldsymbol{Q_1} \quad \boldsymbol{Q_2} \quad \cdots \quad \boldsymbol{Q_p})$$

Multiplying Q from left to the formula 60.

$$\boldsymbol{Q}(\boldsymbol{Q}^{-1}\boldsymbol{C}\boldsymbol{Q}) = \boldsymbol{Q}\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_p \end{pmatrix}$$

$$\boldsymbol{C}\boldsymbol{Q} = (\boldsymbol{Q_1} \quad \boldsymbol{Q_2} \quad \cdots \quad \boldsymbol{Q_n})\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_p \end{pmatrix}$$

$$(\boldsymbol{C}\boldsymbol{Q_1} \quad \boldsymbol{C}\boldsymbol{Q_2} \quad \cdots \quad \boldsymbol{C}\boldsymbol{Q_n}) = (\lambda_1\boldsymbol{Q_1} \ \lambda_2\boldsymbol{Q_2} \cdots \lambda_n\boldsymbol{Q_p})$$

$$\mathbf{C}\boldsymbol{Q_i} = \lambda_i\boldsymbol{Q_i}$$

$\boldsymbol{Q_i}$ is vector. The definition of eigen value and eigen vector is as follow.

$$\mathrm{A}\vec{x} = \lambda\vec{x}$$

From this, we can understand that $\boldsymbol{Q_i}$ is eigen vector for λ_i.

Confirmation using exemplified matrixes in the paragraph of V-2-1. Similarity.

$$\boldsymbol{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{C} = \begin{pmatrix} 8 & 5 & 7 \\ -1 & 2 & -1 \\ -3 & -3 & -2 \end{pmatrix}$$

$$\boldsymbol{P} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\boldsymbol{P}^{-1} = \begin{pmatrix} -1 & 3 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

eigenvector of $\boldsymbol{C}$ as follows

$$\text{Eigen value } (\lambda = 4)\text{: } t_4\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$

$$\text{Eigen value } (\lambda = 3)\text{: } t_3\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\text{Eigen vector } (\lambda = 1)\text{: } t_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Letting $t_4 = 1, t_3 = t_1 = -1$

$$\text{Eigen value } (\lambda = 4)\text{: } \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$

$$\text{Eigen value } (\lambda = 3)\text{: } \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{Eigen vector } (\lambda = 1): \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\boldsymbol{Q_1} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\boldsymbol{Q_2} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$

$$\boldsymbol{Q_3} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\boldsymbol{Q} = (\boldsymbol{Q_1} \quad \boldsymbol{Q_2} \quad \boldsymbol{Q_3}) = \begin{pmatrix} -1 & 3 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\boldsymbol{Q} = \begin{pmatrix} -1 & 3 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\boldsymbol{Q^{-1}} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

We could confirm that the vector which diagonalize a matrix is obtainable by lining up the eigenvectors of the matrix. However, several readers may feel disreputability to the explanation by author. General expression of eigenvector is as follows.

$$\begin{pmatrix} t_3 \\ -t_3 \\ 0 \end{pmatrix}, \begin{pmatrix} 3t_4 \\ -t_4 \\ -t_4 \end{pmatrix}, \begin{pmatrix} t_1 \\ 0 \\ -t_1 \end{pmatrix}$$

This means t_4, t_3 and t_1 is arbitrary real number. We can select any real number for t_4, t_3 and t_1, though the author selected -1 for t_3 and $t_1$and 1 for t_4 for accordance with $\boldsymbol{Q}$. We did not confirm that the vector which diagonalize a matrix is obtainable by lining up any set of the eigenvectors of the matrix.

For clear identification we express t as follow

$$t_4 = b$$

$$t_3 = a$$

$$t_1 = c$$

The set of eigen vector is as follow

$$\begin{pmatrix} -a \\ a \\ 0 \end{pmatrix}, \begin{pmatrix} 3b \\ -b \\ -b \end{pmatrix}, \begin{pmatrix} -c \\ 0 \\ c \end{pmatrix}$$

$$\boldsymbol{Q} = \begin{pmatrix} -a & 3b & -c \\ a & -b & 0 \\ 0 & -b & c \end{pmatrix}$$

$$\boldsymbol{Q^T} = \begin{pmatrix} -a & 3b & -c \\ a & -b & 0 \\ 0 & -b & c \end{pmatrix}' = \begin{pmatrix} -a & a & 0 \\ 3b & -b & -b \\ -c & 0 & c \end{pmatrix}$$

$$|\boldsymbol{Q}| = \begin{vmatrix} -a & 3b & -c \\ a & -b & 0 \\ 0 & -b & c \end{vmatrix} = abc + abc - 3abc = -abc$$

$$\boldsymbol{Q^{-1}} = \frac{-1}{abc}\begin{pmatrix} \begin{vmatrix} -b & -b \\ 0 & c \end{vmatrix} & -\begin{vmatrix} 3b & -b \\ -c & c \end{vmatrix} & \begin{vmatrix} 3b & -b \\ -c & 0 \end{vmatrix} \\ -\begin{vmatrix} a & 0 \\ 0 & c \end{vmatrix} & \begin{vmatrix} -a & 0 \\ -c & c \end{vmatrix} & -\begin{vmatrix} -a & a \\ -c & 0 \end{vmatrix} \\ \begin{vmatrix} a & 0 \\ -b & -b \end{vmatrix} & -\begin{vmatrix} -a & 0 \\ 3b & -b \end{vmatrix} & \begin{vmatrix} -a & a \\ 3b & -b \end{vmatrix} \end{pmatrix}$$

$$= \frac{-1}{abc}\begin{pmatrix} -bc & -2bc & -bc \\ -ac & -ac & -ac \\ -ab & -ab & -2ab \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{a} & \frac{2}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{2}{c} \end{pmatrix}$$

About $\boldsymbol{Q^{-1}CQ}$

$$\boldsymbol{Q^{-1}CQ} = \begin{pmatrix} \frac{1}{a} & \frac{2}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{2}{c} \end{pmatrix}\begin{pmatrix} 8 & 5 & 7 \\ -1 & 2 & -1 \\ -3 & -3 & -2 \end{pmatrix}\begin{pmatrix} a & 3b & c \\ -a & -b & 0 \\ 0 & -b & -c \end{pmatrix}$$

$$\boldsymbol{Q^{-1}C} = \begin{pmatrix} \frac{1}{a} & \frac{2}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{2}{c} \end{pmatrix}\begin{pmatrix} 8 & 5 & 7 \\ -1 & 2 & -1 \\ -3 & -3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{a}(8-2-3) & \frac{1}{a}(5+4-3) & \frac{1}{a}(7-2-2) \\ \frac{1}{b}(8-1-3) & \frac{1}{b}(5+2-3) & \frac{1}{b}(7-1-2) \\ \frac{1}{c}(8-1-6) & \frac{1}{c}(5+2-6) & \frac{1}{c}(7-1-4) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{a} & \frac{6}{a} & \frac{3}{a} \\ \frac{4}{b} & \frac{4}{b} & \frac{4}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{2}{c} \end{pmatrix}$$

$$(\boldsymbol{Q}^{-1}\boldsymbol{C})\boldsymbol{Q} = \begin{pmatrix} \frac{3}{a} & \frac{6}{a} & \frac{3}{a} \\ \frac{4}{b} & \frac{4}{b} & \frac{4}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{2}{c} \end{pmatrix} \begin{pmatrix} a & 3b & c \\ -a & -b & 0 \\ 0 & -b & -c \end{pmatrix}$$

$$= \begin{pmatrix} -3+6 & \frac{b}{a}(-9-6-3) & \frac{c}{a}(-3+0+3) \\ \frac{a}{b}(4-4+0) & (12-4-4) & \frac{c}{b}(4-4) \\ \frac{a}{c}(1-1+0) & \frac{b}{c}(3-1-2) & (1+0-2) \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Coefficients including a, b and c disappear in the process of calculation. From this we can select any eigenvector.

We can understand that p-dimension matrix which has p real number eigenvalue can be diagonalized. However, we still not confirmed whether we cannot diagonalize matrix which has not enough number of real number solution for eigenequation. In other word, we still not confirmed whether we can diagonalize all matrixes.

There are non-diagonal matrixes of which eigenvectors are orthogonal each other. We can make similar matrix from such matrixes. This means we can make such matrixes from other matrixes inversely. However, we can make diagonalized matrix from the matrix by rotation. The author personally thinks that we do not need consider such case. More essentially, we need to consider what we can do in the case the eigenequation has enough solution in real number, or we should consider which matrix can be diagonalized. This is very logical question. However, the author does not discuss this question in detail, because he wants to introduce how to use diagonalization in multivariable analysis.