V-2-3. Spectral decomposition

Several concepts in basic linear algebra are introduced in this text book. The purpose of introduction of basic mathematics is deep understanding of commonly used multivariable analysis techniques. Multivariable analysis is used for simplifying information by summarizing factors to several major factors in many cases. For this purpose, people want to see the data as a superposition of individual impact explained by single independent factor.

Following is a tangible example.
When we look following matrix

$$\boldsymbol{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We decompose the matrix as follow.

$$\boldsymbol{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $1\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are unit vectors which are orthogonal each other. We can recognize structure of data intuitively, because eigen vectors are according with coordinate of matrix.

We denote each unit vector as follow.

$$\boldsymbol{e}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\boldsymbol{D} = 3\boldsymbol{e}_3 + 4\boldsymbol{e}_4 + \boldsymbol{e}_1$$

As shown in above example, we can write each vector by multiplying real to unit vector. This is simplification of dataset. However, we cannot simplify non-diagonal matrixes intuitively. As an example, we cannot consider following matrix as superposition of orthogonal vectors.

$$\begin{pmatrix} 8 & 5 & 7 \\ -1 & 2 & -1 \\ -3 & -3 & -2 \end{pmatrix}$$

In the case of non-diagonal matrix, we cannot simplify the matrix. Of course, we can simplify the matrix after diagonalization of the matrix. However, it is no meaning at this moment, because diagonalized matrix is not the original matrix. Keeping the wish for simplification of matrix as superposition of orthogonal vectors in order to discuss each factor separately, we trace the process of diagonalization step by step

As an example, we diagonalize following matrix.

$$\boldsymbol{A} = \begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$$

1. **Calculation of eigenvalue using eigen equation.**

$$\begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(1-\lambda)(-1-\lambda) - 2 + 9 - 3(2-\lambda) + 2(-1-\lambda) - 3(1-\lambda) = 0$$

$$-\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0$$

$$\lambda = -2, 1, 3$$

2. **Calculation of eigen vector**

Following $\lambda = -2$

$$\begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$4x_1 - 2x_2 + 3x_3 = 0$$

$$x_1 + 3x_2 + x_3 = 0$$

$$x_1 + 3x_2 + x_3 = 0$$

$$x_1 = 11x_2, \quad x_3 = -14x_2$$

Eigenvector

$$t\begin{pmatrix} 11 \\ 1 \\ -14 \end{pmatrix}$$

Following $\lambda = 1$

$$\begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$x_1 - 2x_2 + 3x_3 = 0$$

$$x_1 + x_3 = 0$$

$$x_1 + 3x_2 - 2x_3 = 0$$

$$x_1 = -x_2, \quad x_3 = x_2$$

Eigenvector

$$t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Following $\lambda = 3$

$$\begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 3\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$-x_1 - 2x_2 + 3x_3 = 0$$

$$x_1 - 2x_2 + x_3 = 0$$

$$x_1 + 3x_2 - 4x_3 = 0$$

$$x_1 = x_2 = x_3$$

Eigenvector

$$t\begin{pmatrix}1\\1\\1\end{pmatrix}$$

３．A matrix for orthogonalization of matrix A is

$$\boldsymbol{P} = \begin{pmatrix}11 & -1 & 1\\1 & 1 & 1\\-14 & 1 & 1\end{pmatrix}$$

です。

$$|\boldsymbol{P}| = \begin{vmatrix}11 & -1 & 1\\1 & 1 & 1\\-14 & 1 & 1\end{vmatrix} = 30$$

$$\boldsymbol{P}^T = \begin{pmatrix}11 & 1 & -14\\-1 & 1 & 1\\1 & 1 & 1\end{pmatrix}$$

$$\boldsymbol{P^{-1}} = \frac{1}{|\boldsymbol{P}|}\begin{pmatrix}\begin{vmatrix}1 & 1\\1 & 1\end{vmatrix} & -\begin{vmatrix}-1 & 1\\1 & 1\end{vmatrix} & \begin{vmatrix}-1 & 1\\1 & 1\end{vmatrix}\\-\begin{vmatrix}1 & -14\\1 & 1\end{vmatrix} & \begin{vmatrix}11 & -14\\1 & 1\end{vmatrix} & -\begin{vmatrix}11 & 1\\1 & 1\end{vmatrix}\\\begin{vmatrix}1 & -14\\1 & 1\end{vmatrix} & -\begin{vmatrix}11 & -14\\-1 & 1\end{vmatrix} & \begin{vmatrix}11 & 1\\-1 & 1\end{vmatrix}\end{pmatrix}$$

$$= \frac{1}{30}\begin{pmatrix}0 & 2 & -2\\-15 & 25 & -10\\15 & 3 & 12\end{pmatrix}$$

$$\boldsymbol{P^{-1}AP} = \frac{1}{30}\begin{pmatrix}0 & 2 & -2\\-15 & 25 & -10\\15 & 3 & 12\end{pmatrix}\begin{pmatrix}2 & -2 & 3\\1 & 1 & 1\\1 & 3 & -1\end{pmatrix}\begin{pmatrix}11 & -1 & 1\\1 & 1 & 1\\-14 & 1 & 1\end{pmatrix}$$

$$\frac{1}{30}\begin{pmatrix}0+2-2 & 0+2-6 & 0+2+2\\-30+25-10 & 30+25-30 & -45+25+10\\30+3+12 & -30+3+36 & 45+3-12\end{pmatrix}\begin{pmatrix}11 & -1 & 1\\1 & 1 & 1\\-14 & 1 & 1\end{pmatrix}$$

$$= \begin{pmatrix}0 & -4 & 4\\-15 & 25 & -10\\45 & 9 & 36\end{pmatrix}\begin{pmatrix}11 & -1 & 1\\1 & 1 & 1\\-14 & 1 & 1\end{pmatrix}$$

$$= \frac{1}{30}\begin{pmatrix}0-4-56 & 0-4+4 & 0-4+4\\-165+25+140 & 15+25-10 & -15+25-10\\495+9-504 & -45+9+36 & 45+9+36\end{pmatrix}$$

$$= \frac{1}{30}\begin{pmatrix}-60 & 0 & 0\\0 & 30 & 0\\0 & 0 & 90\end{pmatrix}$$

$$= \begin{pmatrix}-2 & 0 & 0\\0 & 1 & 0\\0 & 0 & 3\end{pmatrix}$$

$$\begin{pmatrix}-2 & 0 & 0\\0 & 1 & 0\\0 & 0 & 3\end{pmatrix} = -2\begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix} + 1\begin{pmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix} + 3\begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{pmatrix}$$

$$\boldsymbol{P^{-1}AP} = -2\begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix} + 1\begin{pmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix} + 3\begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{pmatrix}$$

$$\boldsymbol{P^{-1}AP} = -2\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}^T + 1\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}^T + 3\begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}^T$$

Multiplying $\boldsymbol{P}$ and $\boldsymbol{P^{-1}}$ from left and right respectively

$$\boldsymbol{PP^{-1}APP^{-1}} = \boldsymbol{P}\left(-2\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}^T + 1\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}^T + 3\begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}^T\right)\boldsymbol{P^{-1}}$$

$$\boldsymbol{A} = \boldsymbol{P}\left(-2\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}^T + 1\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}^T + 3\begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}^T\right)\boldsymbol{P^{-1}}$$

Applying law of distribution,

$$\boldsymbol{A} = -2\boldsymbol{P}\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}^T\boldsymbol{P^{-1}} + 1\boldsymbol{P}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}^T\boldsymbol{P^{-1}} + 3\boldsymbol{P}\begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}^T\boldsymbol{P^{-1}}$$

$$= -\frac{2}{30}\begin{pmatrix}11 & -1 & 1\\1 & 1 & 1\\-14 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}^T\begin{pmatrix}0 & 2 & -2\\-15 & 25 & -10\\15 & 3 & 12\end{pmatrix} + \frac{1}{30}\begin{pmatrix}11 & -1 & 1\\1 & 1 & 1\\-14 & 1 & 1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}^T\begin{pmatrix}0 & 2 & -2\\-15 & 25 & -10\\15 & 3 & 12\end{pmatrix}$$

$$+\frac{1}{10}\begin{pmatrix}11 & -1 & 1\\1 & 1 & 1\\-14 & 1 & 1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}^T\begin{pmatrix}0 & 2 & -2\\-15 & 25 & -10\\15 & 3 & 12\end{pmatrix}$$

Here

$$\begin{pmatrix}11 & -1 & 1\\1 & 1 & 1\\-14 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}11\\1\\-14\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 2 & -2\\-15 & 25 & -10\\15 & 3 & 12\end{pmatrix} = \begin{pmatrix}0 & 2 & -2\end{pmatrix}$$

$$\begin{pmatrix}11 & -1 & 1\\1 & 1 & 1\\-14 & 1 & 1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 2 & -2\\-15 & 25 & -10\\15 & 3 & 12\end{pmatrix} = \begin{pmatrix}-15 & 25 & -10\end{pmatrix}$$

$$\begin{pmatrix}11 & -1 & 1\\1 & 1 & 1\\-14 & 1 & 1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 2 & -2\\-15 & 25 & -10\\15 & 3 & 12\end{pmatrix} = \begin{pmatrix}15 & 3 & 12\end{pmatrix}$$

$$\boldsymbol{A} = -2\begin{pmatrix}11\\1\\-14\end{pmatrix}\begin{pmatrix}0 & 2 & -2\end{pmatrix} + \begin{pmatrix}-1\\1\\1\end{pmatrix}\begin{pmatrix}-15 & 25 & -10\end{pmatrix} + 3\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}15 & 3 & 12\end{pmatrix}$$

$$= -2\times\frac{1}{30}\begin{pmatrix}0 & 22 & -22\\0 & 2 & -2\\0 & -28 & 28\end{pmatrix} + \frac{1}{30}\begin{pmatrix}15 & -25 & 10\\-15 & 25 & -10\\-15 & 25 & -10\end{pmatrix} + \frac{1}{10}\begin{pmatrix}15 & 3 & 12\\15 & 3 & 12\\15 & 3 & 12\end{pmatrix}$$

$$= \begin{pmatrix}0+\frac{1}{2}+\frac{3}{2} & \frac{-22}{15}+\frac{-5}{6}+\frac{3}{10} & \frac{22}{15}+\frac{1}{3}+\frac{6}{5}\\ 0-\frac{1}{2}+\frac{3}{2} & -\frac{2}{15}+\frac{5}{6}+\frac{3}{10} & \frac{2}{15}-\frac{1}{3}+\frac{6}{5}\\ 0-\frac{1}{2}+\frac{3}{2} & \frac{28}{15}+\frac{5}{6}+\frac{3}{10} & -\frac{28}{15}-\frac{1}{3}+\frac{6}{5}\end{pmatrix}$$

$$= \begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$$

This is only a confirmation of adequacy of matrix calculation rules. What is noteworthy in the calculation is the line

$$\boldsymbol{A} = -2\boldsymbol{P}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T \boldsymbol{P^{-1}} + 1\boldsymbol{P}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^T \boldsymbol{P^{-1}} + 3\boldsymbol{P}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T \boldsymbol{P^{-1}}$$

This means that when a matrix can be diagonalized as follow

$$\boldsymbol{PA}\mathbf{P^{-1}} = \boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

We can express the matrix as follow

$$\boldsymbol{A} = \lambda_1\boldsymbol{P}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \boldsymbol{P^{-1}} + \lambda_2\boldsymbol{P}\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}^{\boldsymbol{T}} \boldsymbol{P^{-1}} + \cdots + \lambda_n\boldsymbol{P}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}^T \boldsymbol{P^{-1}}$$

We denote the unit vectors as e_i.

$$\boldsymbol{A} = \lambda_1\boldsymbol{Pe_1e_1}^{\boldsymbol{T}}\boldsymbol{P^{-1}} + \lambda_2\boldsymbol{Pe_2e_2}^{\boldsymbol{T}}\boldsymbol{P^{-1}} + \cdots + \lambda_n\boldsymbol{Pe_ne_n}^{\boldsymbol{T}}\boldsymbol{P^{-1}}$$

Apropos of nothing, we consider how we can transform the equation in the case when $\boldsymbol{P^{-1}} = \boldsymbol{P^T}$.

$$\begin{aligned} \boldsymbol{A} &= \lambda_1\boldsymbol{Pe_1e_1}^T\boldsymbol{P^{-1}} + \lambda_2\boldsymbol{Pe_2e_2}^T\boldsymbol{P^{-1}} + \cdots + \lambda_n\boldsymbol{Pe_ne_n}^T\boldsymbol{P^{-1}} \\ &= \lambda_1\boldsymbol{Pe_1e_1}^{\boldsymbol{T}}\boldsymbol{P}^T + \lambda_2\boldsymbol{Pe_2e_2}^T\boldsymbol{P}^T + \cdots + \lambda_n\boldsymbol{Pe_ne_n}^T\boldsymbol{P}^T \\ &= \lambda_1\boldsymbol{Pe_1}(\boldsymbol{Pe_1})^T + \lambda_2\boldsymbol{Pe_2}(\boldsymbol{Pe_2})^T + \cdots + \lambda_n\boldsymbol{Pe_n}(\boldsymbol{Pe_n})^{\boldsymbol{T}} \end{aligned}$$

Some readers may accept the trans formation $\boldsymbol{e_1}^{\boldsymbol{T}}\boldsymbol{P^T} \to (\boldsymbol{Pe_1})^T$ intuitively, Some cannot be accept the transformation without any explanation. We do not need any theoretical explanation. It is acceptable by tracing calculation process of example.

$$\boldsymbol{C} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}, \qquad \boldsymbol{D} = \begin{pmatrix} \alpha & \delta \\ \beta & \varepsilon \\ \gamma & \zeta \end{pmatrix}$$

$$\boldsymbol{C^T} = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}, \qquad \boldsymbol{D^T} = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \zeta \end{pmatrix}$$

$$\boldsymbol{C\ D} = \begin{pmatrix} a\alpha + b\beta + c\gamma & a\delta + b\varepsilon + c\zeta \\ d\alpha + e\beta + f\gamma & d\delta + e\varepsilon + f\zeta \end{pmatrix}$$

$$(\boldsymbol{C\ D})^{\boldsymbol{T}} = \begin{pmatrix} a\alpha + b\beta + c\gamma & d\alpha + e\beta + f\gamma \\ a\delta + b\varepsilon + c\zeta & d\delta + e\varepsilon + f\zeta \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \zeta \end{pmatrix}\begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} = \boldsymbol{D^TC^T}$$

$$\boldsymbol{DC} = \begin{pmatrix} \alpha & \delta \\ \beta & \varepsilon \\ \gamma & \zeta \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} a\alpha + d\delta & b\alpha + e\delta & c\alpha + f\delta \\ a\beta + d\varepsilon & b\beta + e\varepsilon & c\beta + f\varepsilon \\ a\gamma + d\zeta & b\gamma + e\zeta & c\gamma + f\zeta \end{pmatrix}$$

$$(\boldsymbol{DC})^{\boldsymbol{T}} = \begin{pmatrix} a\alpha + d\delta & a\beta + d\varepsilon & a\gamma + d\zeta \\ b\alpha + e\delta & b\beta + e\varepsilon & b\gamma + e\zeta \\ c\alpha + f\delta & c\beta + f\varepsilon & c\gamma + f\zeta \end{pmatrix} = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}\begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \zeta \end{pmatrix} = \boldsymbol{C}^{\boldsymbol{T}}\boldsymbol{D}^{\boldsymbol{T}}$$

When we can transform as follow

$$\boldsymbol{A} = \lambda_1 \boldsymbol{Pe_1}(\boldsymbol{Pe_1})^T + \lambda_2 \boldsymbol{Pe_2}(\boldsymbol{Pe_2})^T + \cdots + \lambda_n \boldsymbol{Pe_n}(\boldsymbol{Pe_n})^T$$

Letting $\boldsymbol{P} = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$

$$\boldsymbol{Pe_1} = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$(\boldsymbol{Pe_1})^T = (a \quad b \quad c)$$

$$\boldsymbol{Pe_1}(\boldsymbol{Pe_1})^T = \begin{pmatrix} a^2 & ab & c \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}$$

$$\boldsymbol{Pe_2} = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$$

$$(\boldsymbol{Pe_2})^T = (d \quad e \quad f)$$

$$\boldsymbol{Pe_2}(\boldsymbol{Pe_2})^{\boldsymbol{T}} = \begin{pmatrix} d^2 & de & df \\ de & e^2 & ef \\ df & ef & f^2 \end{pmatrix}$$

$$\boldsymbol{Pe_3} = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} g \\ h \\ i \end{pmatrix}$$

$$(\boldsymbol{Pe_3})^T = (g \quad h \quad i)$$

$$\boldsymbol{Pe_3}(\boldsymbol{Pe_3})^T = \begin{pmatrix} g^2 & gh & gi \\ gh & h^2 & hi \\ gi & hi & i^2 \end{pmatrix}$$

$$\boldsymbol{A} = \lambda_1 \begin{pmatrix} a \\ b \\ c \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}^T + \lambda_2 \begin{pmatrix} d \\ e \\ f \end{pmatrix}\begin{pmatrix} d \\ e \\ f \end{pmatrix}^T + \lambda_3 \begin{pmatrix} g \\ h \\ i \end{pmatrix}\begin{pmatrix} g \\ h \\ i \end{pmatrix}^T$$

When a matrix has following nature

$$M = M^T$$

the matrix is symmetric matrix.

In another word, when following matrix is symmetric

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

$$a_{ij} = a_{ji}$$

As shown in the calculation of the example, products of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}^T$,

$\begin{pmatrix} d \\ e \\ f \end{pmatrix}\begin{pmatrix} d \\ e \\ f \end{pmatrix}^T$ and $\begin{pmatrix} g \\ h \\ i \end{pmatrix}\begin{pmatrix} g \\ h \\ i \end{pmatrix}^T$ are symmetric matrix.

And the sum of the symmetric matrixes is symmetric matrix. So, A is symmetric.

From this, we can say that symmetric matrixes can be decomposed to sum of symmetric matrixes, if symmetric matrixes generally have nature that $P^{-1} = P^T$.

At first, we consider the necessary condition of $P^{-1} = P^T$

The equation of $P^T = P^{-1}$ means

$$P^T P = I$$

We consider this condition from simple easy example.

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$P^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$P^T P = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

From this equation

$$a^2 + c^2 = 1 \qquad \text{i}$$

$$b^2 + d^2 = 1 \qquad \text{ii}$$

$$ab + cd = 0 \qquad \text{iii}$$

We can solve this simultaneous equation. However, when we consider

$$P = (P_1 \quad P_2)$$

where

$$P_1 = \begin{pmatrix} a \\ c \end{pmatrix}$$

$$\boldsymbol{P_2} = \begin{pmatrix} b \\ d \end{pmatrix}$$

The equation of i is $|\boldsymbol{P_1}|^2$ and the equation ii is $|\boldsymbol{P_2}|^2$. The equation iii is inner product of $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$

$$\boldsymbol{P_1 P_2 = 0}$$

From this, the necessary condition is $\boldsymbol{P_1}$ **and** $\boldsymbol{P_2}$ are unit vectors and are orthogonal each other

In the case of 3×3 square matrix

$$\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a^2+d^2+g^2 & ab+de+hg & ac+fd+ig \\ ba+de+hg & b^2+e^2+h^2 & cb+ef+ih \\ ac+fd+ig & cb+ef+ih & c^2+f^2+i^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$a^2+d^2+g^2 = 1$$

$$b^2+e^2+h^2 = 1$$

$$c^2+f^2+i^2 = 1$$

$$ab+de+hg = 0$$

$$ac+fd+ig = 0$$

$$cb+ef+ih = 0$$

This means the vectors which is the element of $\boldsymbol{P}$ are unit vectors, and they are orthogonal each other. Originally the vectors composing $\boldsymbol{P}$ is originally eigenvectors of matrix $\boldsymbol{A}$. From this we can conclude that when the eigenvectors of a symmetric matrix is orthogonal each other $\boldsymbol{P^{-1}} = \boldsymbol{P}^T$

Next, we verify that eigenvectors of symmetric matrixes are orthogonal each other. $\boldsymbol{x_1}, \boldsymbol{x_2}$ are eigenvector of matrix $\boldsymbol{A}$. From the definition of eigenvalue and eigenvector.

$$\boldsymbol{A x_1} = \lambda_1 \boldsymbol{x_1} \qquad \text{i}$$

$$\boldsymbol{A x_2} = \lambda_2 \boldsymbol{x_2} \qquad \text{ii}$$

Transpose i

$$(\boldsymbol{A x_1})^T = \lambda_1 \boldsymbol{x_1}^T$$

$$\boldsymbol{x_1}^T \boldsymbol{A}^T = \lambda_1 \boldsymbol{x_1}^T \qquad \text{i'}$$

When matrix $\boldsymbol{A}$ is symmetric matrix

$$\boldsymbol{A}^T = \boldsymbol{A}$$

Put this in i'

$$\boldsymbol{x_1}^T \boldsymbol{A} = \lambda_1 \boldsymbol{x_1}^T$$

Multiply $\boldsymbol{x_2}$ to both sides from right

$$\boldsymbol{x_1}^T \boldsymbol{A}\boldsymbol{x_2} = \lambda_1 \boldsymbol{x_1}^T \boldsymbol{x_2}$$

Put ii to left side of upper equation

$$\boldsymbol{x_1}^T \lambda_2 \boldsymbol{x_2} = \lambda_1 \boldsymbol{x_1}^T \boldsymbol{x_2}$$

$$\lambda_2 \boldsymbol{x_1}^T \boldsymbol{x_2} = \lambda_1 \boldsymbol{x_1}^T \boldsymbol{x_2}$$

Transpose right side to left side

$$(\lambda_2 - \lambda_1)\boldsymbol{x_1}^T \boldsymbol{x_2} = 0$$

From this

$$\lambda_2 - \lambda_1 = 0$$

or

$$\boldsymbol{x_1}^T \boldsymbol{x_2} = 0$$

In the case the eigen equation has no multiple root

$$\lambda_2 - \lambda_1 \neq 0$$

Then

$$\boldsymbol{x_1}^T \boldsymbol{x_2} = 0$$

Product of $\boldsymbol{x_1}^T \boldsymbol{x_2}$ is inner product of $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$. When $\boldsymbol{x_1}^T \boldsymbol{x_2}$ =0, $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ are orthogonal each other

We must consider the case of having multiple root separately. In the case of double root, the eigenvectors following the double roots is exist in a plane which is orthogonal to other eigenvectors. We can select any unit vector on the plane for a double root and can find orthogonal unit vector for the other double root. This method is named Gram-Schmidt orthogonalization. In the case of higher-order multiple root, the space the eigenvectors following the multiple root exist orthogonal space to other eigenvectors. We can use Gram-Schmidt orthogonalization.

Another issue we should consider is the case when the eigen equation has no solution. However, all symmetric matrix has solution. From this we can conclude that when a matrix is symmetric matrix, we can decompose the matrix as follow

$$\boldsymbol{A} = \boldsymbol{\Lambda} \boldsymbol{e}\boldsymbol{e}^T$$

($\boldsymbol{e}$ is matrix composed from unit vector of eigenvector, and

$$\boldsymbol{\Lambda} \text{ (lambda)} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

More applicable expression is as follows

$$\boldsymbol{A} = \lambda_1 \boldsymbol{e_1}\boldsymbol{e_1}^T + \lambda_2 \boldsymbol{e_2}\boldsymbol{e_2}^T + \cdots + \lambda_p \boldsymbol{e_p}\boldsymbol{e_p}^T$$

or

$$A = \sum_{i=1}^{p} \lambda_i \boldsymbol{e_i} \boldsymbol{e_i}^T$$

$\boldsymbol{e_i}$: unit vector of eigenvector,

$$\boldsymbol{e_i} \perp \boldsymbol{e_j}$$

Formula 61

We call decomposing of symmetric matrix to sum of symmetric matrixes produced from eigenvectors which orthogonal each other by this method as spectral decomposition.

For understanding of specific protocol of spectral decomposition, we implement specific decomposition of following matrix.

$$\boldsymbol{A} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 5 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

$\boldsymbol{A}$ is a symmetric matrix.

1. Eigen equation

$$\begin{vmatrix} (3-\lambda) & 1 & -1 \\ 1 & (5-\lambda) & -1 \\ -1 & -1 & (3-\lambda) \end{vmatrix} = 0$$

$$(3-\lambda)(5-\lambda)(3-\lambda) + 1 + 1 - (3-\lambda) - (5-\lambda) - (3-\lambda) = 0$$

$$36 - 36\lambda + 11\lambda^2 - \lambda^3 = 0$$

$$(\lambda - 6)(\lambda - 3)(\lambda - 2) = 0$$

$$\lambda_1 = 6, \lambda_2 = 3, \lambda_3 = 2$$

2. Eigenvector

Eigenvector belonging to $\lambda_1 = 6$

$$\begin{pmatrix} 3 & 1 & -1 \\ 1 & 5 & -1 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 6 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$3x_1 + x_2 - x_3 = 6x_1$ i

$x_1 + 5x_2 - x_3 = 6x_2$ ii

$-x_1 - x_2 + 3x_3 = 6x_3$ iii

$-3x_1 + x_2 - x_3 = 0$ i'

$x_1 - x_2 - x_3 = 0$ ii'

$-x_1 - x_2 - 3x_3 = 0$ iii'

$-2x_1 - 2x_3 = 0$ i'+ii'

$-2x_2 - 4x_3 = 0$ ii'+iii'

$$x_1 = -x_3$$

$$x_2 = -2x_3$$

$$\boldsymbol{e_1} = \mathrm{t}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Eigenvector belonging to $\lambda_2 = 3$

$$\begin{pmatrix} 3 & 1 & -1 \\ 1 & 5 & -1 \\ -1 & -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 3\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$3x_1 + x_2 - x_3 = 3x_1$ i

$x_1 + 5x_2 - x_3 = 3x_2$ ii

$-x_1 - x_2 + 3x_3 = 3x_3$ iii

$x_2 - x_3 = 0$ i'

$x_1 + 2x_2 - x_3 = 0$ ii'

$-x_1 - x_2 = 0$ iii'

$$x_1 = -x_2$$

$$x_2 = x_3$$

$$\boldsymbol{e_2} = \mathrm{t}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

Eigenvector belonging to $\lambda_3 = 2$

$$\begin{pmatrix} 3 & 1 & -1 \\ 1 & 5 & -1 \\ -1 & -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$3x_1 + x_2 - x_3 = 2x_1$ i

$x_1 + 5x_2 - x_3 = 2x_2$ ii

$-x_1 - x_2 + 3x_3 = 2x_3$ iii

$x_1 + x_2 - x_3 = 0$ i'

$x_1 + 3x_2 - x_3 = 0$ ii'

$-x_1 - x_2 + x_3 = 0$ iii'

$-2x_2 = 0$ i' − ii'

$$x_2 = 0$$

$x_1 - x_3 = 0$ substitution of $x_2 = 0$ in i'

$$x_1 = x_3$$

$$\boldsymbol{e_2} = \mathrm{t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Unit vectors of each eigen vector

$$\boldsymbol{e_1} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \end{pmatrix}, \quad \boldsymbol{e_2} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix}, \boldsymbol{e_3} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Result of spectral decomposition.

$$\boldsymbol{A} = 6\boldsymbol{e_1}\boldsymbol{e_1}^T + 3\boldsymbol{e_2}\boldsymbol{e_2}^{\boldsymbol{T}} + 2\boldsymbol{e_n}\boldsymbol{e_n}^T$$

$$\boldsymbol{A} = 6\begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \end{pmatrix}^T + 3\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix}^T + 2\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}^T$$

Confirmation

First term of right side

$$6\begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \end{pmatrix}^T = 6\begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{pmatrix}$$

$$= 6\begin{pmatrix} \frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{4}{6} & -\frac{1}{3} \\ -\frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix}$$

Second term of right side

$$3\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix}^T = 3\begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

$$3\begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

Third term of right side

$$2\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}^T = 2\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= 2\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Right side

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 5 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

Quasi spectral decomposition

Strictly, spectra decomposition means decomposition of symmetric matirix. Asymmetric matrixes can also be decomposed to sum of products of eigenvalue and matrix similarly as spectral decomposition of symmetric matrix. This decomposition is included in spectral decomposition in several text books. However, it cannot be expressed only by eigenvectors and result of decomposition is different from spectral decomposition, because we cannot expect complete orthogonality among eigenvectors of asymmetric. The author does not know how we call the decomposition of asymmetric matrix. We call the decomposition asymmetric matrix as quasi-spectral decomposition.

Let presume asymmetric matrix $\boldsymbol{A}$ can be diagonalized by $\boldsymbol{P}$

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

$$\boldsymbol{P}\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}\boldsymbol{P}^{-1} = \boldsymbol{A}$$

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1} = \boldsymbol{P}\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}\boldsymbol{P}^{-1}$$

Put $\boldsymbol{P} := \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}$ and $\boldsymbol{P}^{-1} = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}$

$$\begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}\begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 p_{11} & \lambda_2 p_{12} & \cdots & \lambda_n p_{1n} \\ \lambda_1 p_{21} & \lambda_2 p_{22} & \cdots & \lambda_n p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 p_{n1} & \lambda_2 p_{n2} & \cdots & \lambda_n p_{nn} \end{pmatrix}\begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}$$

$$
= \begin{pmatrix} \lambda_1 p_{11} q_{11} + \lambda_2 p_{12} q_{21} + \cdots + \lambda_n p_{1n} q_{n1} & \lambda_1 p_{11} q_{12} + \lambda_2 p_{12} q_{22} + \cdots + \lambda_n p_{1n} q_{n2} & \cdots & \lambda_1 p_{11} q_{1n} + \lambda_2 p_{12} q_{2n} + \cdots + \lambda_n p_{1n} q_{nn} \\ \lambda_1 p_{21} q_{11} + \lambda_2 p_{22} q_{21} + \cdots + \lambda_n p_{2n} q_{n1} & \lambda_1 p_{21} q_{12} + \lambda_2 p_{22} q_{22} + \cdots + \lambda_n p_{2n} q_{n2} & \cdots & \lambda_1 p_{21} q_{1n} + \lambda_2 p_{22} q_{2n} + \cdots + \lambda_n p_{2n} q_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 p_{n1} q_{11} + \lambda_2 p_{n2} q_{21} + \cdots + \lambda_n p_{nn} q_{n1} & \lambda_1 p_{n1} q_{12} + \lambda_2 p_{n2} q_{22} + \cdots + \lambda_n p_{nn} q_{n2} & \cdots & \lambda_1 p_{n1} q_{1n} + \lambda_2 p_{n2} q_{2n} + \cdots + \lambda_n p_{nn} q_{nn} \end{pmatrix}
$$

$$
= \lambda_1 \begin{pmatrix} p_{11}q_{11} & p_{11}q_{12} & \cdots & p_{11}q_{1n} \\ p_{21}q_{11} & p_{21}q_{12} & \cdots & p_{21}q_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1}q_{11} & p_{n1}q_{12} & \cdots & p_{n1}q_{1n} \end{pmatrix} + \lambda_2 \begin{pmatrix} p_{12}q_{21} & p_{12}q_{22} & \cdots & p_{12}q_{2n} \\ p_{22}q_{21} & p_{22}q_{22} & \cdots & p_{22}q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n2}q_{21} & p_{n2}q_{22} & \cdots & p_{n2}q_{2n} \end{pmatrix} + \cdots + \lambda_n \begin{pmatrix} p_{1n}q_{n1} & p_{1n}q_{n2} & \cdots & p_{1n}q_{nn} \\ p_{2n}q_{n1} & p_{2n}q_{n2} & \cdots & p_{2n}q_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ p_{nn}q_{n1} & p_{nn}q_{n2} & \cdots & p_{nn}q_{nn} \end{pmatrix}
$$

$$
\begin{pmatrix} p_{11}q_{11} & p_{11}q_{12} & \cdots & p_{11}q_{1n} \\ p_{21}q_{11} & p_{21}q_{12} & \cdots & p_{21}q_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1}q_{11} & p_{n1}q_{12} & \cdots & p_{n1}q_{1n} \end{pmatrix} = \begin{pmatrix} p_{11} & 0 & \cdots & 0 \\ p_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}
$$

$$
\begin{pmatrix} p_{12}q_{21} & p_{12}q_{22} & \cdots & p_{12}q_{2n} \\ p_{22}q_{21} & p_{22}q_{22} & \cdots & p_{22}q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n2}q_{21} & p_{n2}q_{22} & \cdots & p_{n2}q_{2n} \end{pmatrix} = \begin{pmatrix} 0 & p_{12} & \cdots & 0 \\ 0 & p_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & p_{n2} & \cdots & 0 \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}
$$

$$
\begin{pmatrix} p_{1n}q_{n1} & p_{1n}q_{n2} & \cdots & p_{1n}q_{nn} \\ p_{2n}q_{n1} & p_{2n}q_{n2} & \cdots & p_{2n}q_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ p_{nn}q_{n1} & p_{nn}q_{n2} & \cdots & p_{nn}q_{nn} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & p_{1n} \\ 0 & 0 & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_{nn} \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}
$$

$$
\therefore \boldsymbol{A} = \lambda_1 \begin{pmatrix} p_{11} & 0 & \cdots & 0 \\ p_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & p_{12} & \cdots & 0 \\ 0 & p_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & p_{n2} & \cdots & 0 \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix} + \cdots
$$

$$
+ \lambda_n \begin{pmatrix} 0 & 0 & \cdots & p_{1n} \\ 0 & 0 & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_{nn} \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}
$$

This is quasi spectral decomposition.

An example of quasi spectral decomposition (This example is used in V-2-2. Diagonalization)

$$
\boldsymbol{A} = \begin{pmatrix} 8 & 5 & 7 \\ -1 & 2 & -1 \\ -3 & -3 & -2 \end{pmatrix}
$$

Change to unit vector

$$
\boldsymbol{P} = \begin{pmatrix} \frac{3}{\sqrt{11}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{11}} & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{11}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}
$$

$$\boldsymbol{P^{-1}} = \begin{pmatrix} \sqrt{11} & \sqrt{11} & \sqrt{11} \\ -\sqrt{2} & -2\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & -2\sqrt{2} \end{pmatrix}$$

$$\boldsymbol{P_1} = \begin{pmatrix} \frac{3}{\sqrt{11}} & 0 & 0 \\ -\frac{1}{\sqrt{11}} & 0 & 0 \\ -\frac{1}{\sqrt{11}} & 0 & 0 \end{pmatrix}, \quad \boldsymbol{P_2} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{P_3} = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\boldsymbol{P} = \boldsymbol{P_1} + \boldsymbol{P_2} + \boldsymbol{P_3}$$

$$\boldsymbol{P_1P^{-1}} = \begin{pmatrix} \frac{3}{\sqrt{11}} & 0 & 0 \\ -\frac{1}{\sqrt{11}} & 0 & 0 \\ -\frac{1}{\sqrt{11}} & 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{11} & \sqrt{11} & \sqrt{11} \\ -\sqrt{2} & -2\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & -2\sqrt{2} \end{pmatrix} = \begin{pmatrix} 3 & 3 & 3 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

$$\boldsymbol{P_2P^{-1}} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{11} & \sqrt{11} & \sqrt{11} \\ -\sqrt{2} & -2\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & -2\sqrt{2} \end{pmatrix} = \begin{pmatrix} -1 & -2 & -1 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{P_3P^{-1}} = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{11} & \sqrt{11} & \sqrt{11} \\ -\sqrt{2} & -2\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & -2\sqrt{2} \end{pmatrix} = \begin{pmatrix} -1 & -1 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\boldsymbol{A} = 3\boldsymbol{P_1P^{-1}} + 4\boldsymbol{P_2P^{-1}} + 1\boldsymbol{P_3P^{-1}}$$

Confirmation

$$\boldsymbol{A} = \begin{pmatrix} 8 & 5 & 7 \\ -1 & 2 & -1 \\ -3 & -3 & -2 \end{pmatrix}$$

$$3\boldsymbol{P_1Q} + 4\boldsymbol{P_2Q} + 1\boldsymbol{P_3Q}$$

$$= 4\begin{pmatrix} 3 & 3 & 3 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} + 3\begin{pmatrix} -1 & -2 & -1 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} + +1\begin{pmatrix} -1 & -1 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 12-3-1 & 12-6-1 & 12-3-2 \\ -4+3+0 & -4+6+0 & -4+3+0 \\ -4+0+1 & -4+0+1 & -4+0+2 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 5 & 7 \\ -1 & 2 & -1 \\ -3 & -3 & -2 \end{pmatrix}$$

$$\boldsymbol{A^2} = \begin{pmatrix} 8 & 5 & 7 \\ -1 & 2 & -1 \\ -3 & -3 & -2 \end{pmatrix} \begin{pmatrix} 8 & 5 & 7 \\ -1 & 2 & -1 \\ -3 & -3 & -2 \end{pmatrix} = \begin{pmatrix} 64-5-21 & 40+10-21 & 56-5-14 \\ -8-2+3 & -5+4+3 & -7-2+2 \\ -24+3+6 & -15-6+6 & -21+3+4 \end{pmatrix}$$

$$= \begin{pmatrix} 38 & 29 & 37 \\ -7 & 2 & -7 \\ -15 & -15 & -14 \end{pmatrix}$$

$$4^2 \boldsymbol{P_1 Q} + 3^2 \boldsymbol{P_2 Q} + 1^2 \boldsymbol{P_3 Q}$$

$$= 16 \begin{pmatrix} 3 & 3 & 3 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} + 9 \begin{pmatrix} -1 & -2 & -1 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} + 1 \begin{pmatrix} -1 & -1 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 48 - 9 - 1 & 48 - 18 - 1 & 48 - 9 - 2 \\ -16 + 9 + 0 & -16 + 18 + 0 & -16 + 9 + 0 \\ -16 + 0 + 1 & -16 + 0 + 1 & -16 + 0 + 2 \end{pmatrix}$$

$$= \begin{pmatrix} 38 & 29 & 37 \\ -7 & 2 & -7 \\ -15 & -15 & -14 \end{pmatrix}$$

We can calculate power of Asymmetric matrix by following formula when the matrix can be diagonalized.

$$\boldsymbol{A}^m = \sum_{\boldsymbol{k=1}}^{\boldsymbol{n}} {\lambda_k}^m \boldsymbol{P_k P^{-1}}$$