V-2-4. Quadratic form

V-2-4-1. Symmetric matrix and Quadratic form

We denote following form of formulas as quadratic form.

$$ax_1{}^2 + bx_2{}^2 + cx_3{}^2 + dx_1x_2 + ex_2x_3 + fx_3x_1$$

Formula in quadratic form is composed only from second order terms. When we generalize this rule of terminology, we must say following formula as liner form or primary order form

$$\mathrm{a}x_1 + bx_2 + cx_3$$

the formula can be expressed as follow using matrix.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}^T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Quadratic form is able to express as follow

$$(x_1 \quad x_2 \quad x_3)\begin{pmatrix} a & d & f \\ 0 & b & e \\ 0 & 0 & c \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Confirmation

$$(x_1 \quad x_2 \quad x_3)\begin{pmatrix} a & d & f \\ 0 & b & e \\ 0 & 0 & c \end{pmatrix} = (ax_1 \quad dx_1 + bx_2 + \quad fx_1 + ex_2 + cx_3)$$

$$(ax_1 \quad dx_1 + bx_2 + \quad fx_1 + ex_2 + cx_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= ax_1{}^2 + bx_2{}^2 + cx_3{}^2 + dx_1x_2 + ex_2x_3 + fx_3x_1$$

This is natural and is not false, though we want to express as follow

$$(x_1 \quad x_2 \quad x_3)\begin{pmatrix} a & \frac{d}{2} & \frac{f}{2} \\ \frac{d}{2} & b & \frac{e}{2} \\ \frac{f}{2} & \frac{e}{2} & c \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Because, the matrix is symmetric.

Confirmation

$$(x_1 \quad x_2 \quad x_3)\begin{pmatrix} a & \frac{d}{2} & \frac{f}{2} \\ \frac{d}{2} & b & \frac{e}{2} \\ \frac{f}{2} & \frac{e}{2} & c \end{pmatrix} = \left(ax_1 + \frac{d}{2}x_2 + \frac{f}{2}x_3 \quad \frac{d}{2}x_1 + bx_2 + \frac{e}{2}x_3 \quad \frac{f}{2}x_1 + \frac{e}{2}x_2 + cx_3\right)$$

$$\left(ax_1 + \frac{d}{2}x_2 + \frac{f}{2}x_3 \quad \frac{d}{2}x_1 + bx_2 + \frac{e}{2}x_3 \quad \frac{f}{2}x_1 + \frac{e}{2}x_2 + cx_3\right)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \mathrm{a}x_1{}^2 + \frac{d}{2}x_1x_2 + \frac{f}{2}x_1x_3 + \frac{d}{2}x_1x_2 + bx_2{}^2 + \frac{e}{2}x_1x_3 + \frac{f}{2}x_1x_3 + \frac{e}{2}x_2x_3 + cx_3{}^2$$

$$= a{x_1}^2 + b{x_2}^2 + c{x_3}^2 + dx_1x_2 + ex_2x_3 + fx_3x_1$$

Generally, we can express quadratic form as follow using symmetric matrix.

$$(x_1 \quad x_2 \quad \cdots \quad x_p)\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{ip} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{p1} & \cdots & a_{pj} & \cdots & a_{pp} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}$$

When we denote vector $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}$ as $\boldsymbol{x}$, and the matrix $\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{ip} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{p1} & \cdots & a_{pj} & \cdots & a_{pp} \end{pmatrix}$ as $\boldsymbol{A}$

Quadratic form can be express as follow.

$$\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$$

V-2-4-2. Diagonalization of quadratic form

Quadratic form has figural properties, such as ellipse, parabola and hyperbola. However, it is difficult to visualize the figures in multidimensional space. The author shows several simple examples in 2-dimensional plane.

First example is $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

Diagonalization

$$\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 0$$

$$(\lambda - 1)^2 = 4$$

$$\lambda = 3, -1$$

$$x_1 + 2x_2 = 3x_1$$

$$2x_1 + x_2 = 3x_2$$

$$x_1 = x_2$$

$$\boldsymbol{e}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$x_1 + 2x_2 = -x_1$$

$$2x_1 + x_2 = -x_2$$

$$x_1 = -x_2$$

$$\boldsymbol{e}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} + \sqrt{2} & \sqrt{2} + \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} - \sqrt{2} & \sqrt{2} - \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + 1 + 1 + \frac{1}{2} & \frac{1}{2} + 1 - 1 - \frac{1}{2} \\ \frac{1}{2} - 1 + 1 - \frac{1}{2} & \frac{1}{2} - 1 - 1 + \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

Spectral decomposition

$$\lambda_1 \boldsymbol{e}_1 \boldsymbol{e}_1{}^T = 3 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = 3 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\lambda_2 \boldsymbol{e}_2 \boldsymbol{e}_2{}^T = -1 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = -\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \lambda_1 \boldsymbol{e}_1 \boldsymbol{e}_1{}^T + \lambda_2 \boldsymbol{e}_2 \boldsymbol{e}_2{}^T = 3 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} - 1 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= 3 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Using result of spectral decomposition, we make quadratic form

$$\boldsymbol{x}^T \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \boldsymbol{x} = \lambda_1 \boldsymbol{x}^T \boldsymbol{e}_1 \boldsymbol{e}_1{}^T \boldsymbol{x} + \lambda_2 \boldsymbol{x}^T \boldsymbol{e}_2 \boldsymbol{e}_2{}^T \boldsymbol{x}$$

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\lambda_1 \boldsymbol{x}^T \boldsymbol{e}_1 {\boldsymbol{e}_1}^T \boldsymbol{x} = 3(x_1 \quad x_2)\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3\begin{pmatrix} \frac{x_1 + x_2}{2} & \frac{x_1 + x_2}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= 3\begin{pmatrix} \frac{x_1 + x_2}{2} & \frac{x_1 + x_2}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{3}{2}({x_1}^2 + 2x_1x_2 + {x_1}^2) = 3\left(\frac{x_1 + x_2}{\sqrt{2}}\right)^2$$

$$\lambda_2 \boldsymbol{x}^T \boldsymbol{e}_2 {\boldsymbol{e}_2}^T \boldsymbol{x} = -(x_1 \quad x_2)\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} \frac{x_1 - x_2}{2} & \frac{-x_1 + x_2}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= -\frac{1}{2}({x_1}^2 - 2x_1x_2 + {x_1}^2) = -\left(\frac{x_2 - x_1}{\sqrt{2}}\right)^2$$

$$\therefore \quad \lambda_1 \boldsymbol{x}^T \boldsymbol{e}_1 {\boldsymbol{e}_1}^T \boldsymbol{x} + \lambda_2 \boldsymbol{x}^T \boldsymbol{e}_2 {\boldsymbol{e}_2}^T \boldsymbol{x} = 3\left(\frac{x_1 + x_2}{\sqrt{2}}\right)^2 - \left(\frac{x_2 - x_1}{\sqrt{2}}\right)^2 \qquad \text{i}$$

On the other hand

$$\boldsymbol{x}^T \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \boldsymbol{x} = (x_1 \quad x_2)\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1 + 2x_2 \quad 2x_1 + x_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = {x_1}^2 + 4x_1x_2 + {x_2}^2 \qquad \text{ii}$$

From i and ii

$${x_1}^2 + 4x_1x_2 + {x_2}^2 = 3\left(\frac{x_1 + x_2}{\sqrt{2}}\right)^2 - \left(\frac{x_2 - x_1}{\sqrt{2}}\right)^2$$

This transformation brings us following information. When we denote each terms of right side as follow

$$x_1 + x_2 = X_1$$
$$x_2 - x_1 = X_2$$
$${x_1}^2 + 4x_1x_2 + {x_2}^2 = \frac{{X_1}^2}{\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2} - \frac{{X_2}^2}{\sqrt{2}^2}$$

Figure 55 shows the relation between axes of two coordinate before and after transformation. Red allows are eigenvectors of original matrix (heavy line denotes unit vector.). From this figure, we can understand that the meaning of diagonalization of quadratic form (Symmetric matrix) is redrawing data distribution taking eigenvectors (heavy red allows) as basis of space.

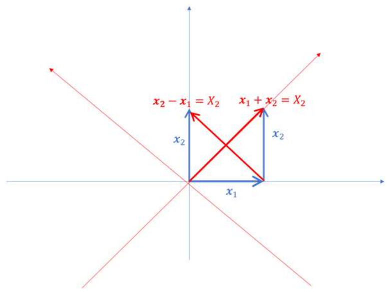

Fig. 55. Eigenvectors in original coordinate

Quadratic form itself has no figure as it is, though the locus of variables shows particular shape (conic curve), when quadratic form has a real value. Conic curves are shapes of cross-sections of circular cone cut by various plane and are categorized by eccentricity to circle, ellipse, parabola and hyperbola. (Eccentricity is are ratio of distance from directrix and distance from focus to the point.) One of the conic curves is hyperbola. Simpler and practical definition of hyperbola is as follow. Hyperbola is locus point where the absolute value of difference of the distances from two focuses is constant.

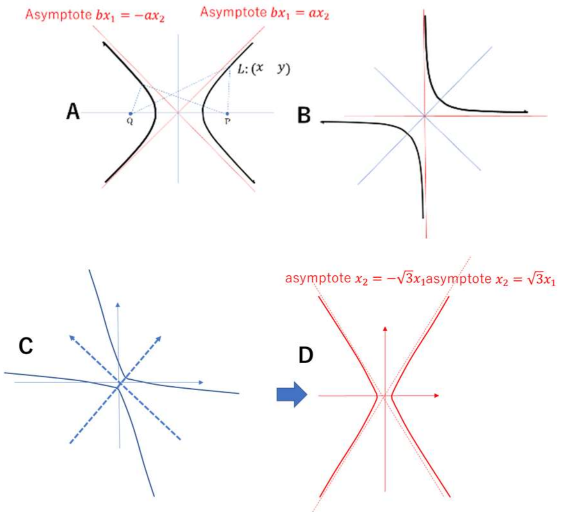

Fig 56. Various hyperbolas

Figure 56 shows examples of hyperbola.

$$\mathrm{P}:\ (f \quad 0)$$

$$\mathrm{Q}:\ (f \quad 0)$$

$$|L\mathrm{Q}| - |L\mathrm{P}| = |2a| = \pm 2a$$

$$|L\mathrm{Q}| = \pm 2a + |L\mathrm{P}|$$

$$|LQ|^2 = 4a^2 \pm 4a|L\mathrm{P}| + |L\mathrm{P}|^2$$

$$|L\mathrm{P}| = \sqrt{(x-f)^2 + (y-0)^2}$$

$$|L\mathrm{Q}| = \sqrt{(x+f)^2 + (y-0)^2}$$

$$(x+f)^2 + y^2 = 4a^2 \pm 4a\sqrt{(x-f)^2 + y^2} + (x-f)^2 + y^2$$

$$4fx - 4a^2 = 4a\sqrt{(x-f)^2 + y^2}$$

$$fx - a^2 = a\sqrt{(x-f)^2 + y^2}$$

$$(fx - a^2)^2 = a^2(x-f)^2 + a^2y^2$$

$$f^2x^2 - 2c^2fx + a^2 = a^2x^2 - 2a^2fx + a^2f^2 + a^2y^2$$

$$(f^2 - a^2)x^2 - a^2y^2 = a^2f^2 - a^4$$

$$(f^2 - a^2)x^2 - a^2y^2 = a^2(f^2 - a^2)$$

$$\frac{x^2}{a^2} - \frac{y^2}{(f^2 - a^2)} = 1$$

Let $(f^2 - a^2) = b^2$

$$f^2 = a^2 + b^2$$

$$f = \pm\sqrt{a^2 + b^2}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

This is the general form of formula of hyperbola. From the general form,

$$\frac{x^2}{a^2} - 1 = \frac{y^2}{b^2}$$

$$1 - \frac{a^2}{x^2} = \frac{a^2y^2}{b^2x^2}$$

$$\lim_{x\to\pm\infty}\left(1 - \frac{a^2}{x^2}\right) = 1$$

$$\lim_{x\to\pm\infty}\frac{a^2y^2}{b^2x^2} = 1$$

$$\lim_{x\to\pm\infty}\frac{ay}{bx} = \pm 1$$

This means that the locus gradually closes the line of $\frac{ay}{bx} = \pm 1$ with the increase of x. From this, we call the line of $ay = bx$ and $ay = -bx$ as asymptote (red line in graph A and B)

A in figurer 56 shows $x^2 - y^2 = 1$. We rotate graph of A, $\frac{\pi}{4}$ to anticlockwise direction. Equation of the rotation is as follow

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\theta = \frac{\pi}{4}$$

$$\cos\theta = \frac{1}{\sqrt{2}}, \qquad \sin\theta = \frac{1}{\sqrt{2}}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$$

$$x = \frac{1}{\sqrt{2}}(X + Y)$$

$$y = \frac{1}{\sqrt{2}}(-X + Y)$$

$$x^2 - y^2 = \frac{1}{2}((X + Y)^2 - (-X + Y)^2) = 2XY$$

$$2XY = 1$$

$$Y = \frac{1}{2X}$$

This is formula of inverse proportion. Formula of inverse proportion is a formula of hyperbola and X axis and Y axis are asymptote. Generally, the focuses do not exist on the horizontal axis. We can rotate the locus of quadratic form by diagonalization. Graph C in figure 56 is locus following matrix equation.

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1$$

From this we can obtain following quadratic equation.

$$x_1{}^2 + 4x_1x_2 + x_2{}^2 = 1$$
$$(x_2 + 2x_1)^2 - 3x_1{}^2 = 1$$
$$(x_2 + 2x_1)^2 = 1 + 3x_1{}^2$$
$$x_2 + 2x_1 = \pm\sqrt{1 + 3x_1{}^2}$$
$$x_2 = -2x_1 \pm \sqrt{1 + 3x_1{}^2}$$

$$\frac{x_2}{x_1} = -2 \pm \sqrt{\frac{1}{x^2} + 3}$$

$$\lim_{x_1 \to \pm\infty} \left(-2 \pm \sqrt{\frac{1}{x^2} + 3} \right) = -2 \pm \sqrt{3}$$

$$\lim_{x_1 \to \pm\infty} \frac{x_2}{x_1} = -2 \pm \sqrt{3}$$

Asymptote are $x_2 = -(2 + \sqrt{3})x_1$ and $x_2 = -(2 - \sqrt{3})x_1$

We rotate this locus -

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ -(2 + \sqrt{3}) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} + \frac{2 + \sqrt{3}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} - \frac{2 + \sqrt{3}}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{3 + \sqrt{3}}{\sqrt{2}} \\ \frac{-1 - \sqrt{3}}{\sqrt{2}} \end{pmatrix}$$

$$\frac{X_1}{X_2} = \frac{3 + \sqrt{3}}{-1 - \sqrt{3}} = \frac{-(3 + \sqrt{3})(\sqrt{3} - 1)}{(\sqrt{3} + 1)(\sqrt{3} - 1)} = \frac{-(3 - 3 - 2\sqrt{3})}{3 - 1} = \sqrt{3}$$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ -(2-\sqrt{3}) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} + \frac{2-\sqrt{3}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} - \frac{2-\sqrt{3}}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{3-\sqrt{3}}{\sqrt{2}} \\ \frac{-1+\sqrt{3}}{\sqrt{2}} \end{pmatrix}$$

$$\frac{X_1}{X_2} = \frac{3-\sqrt{3}}{-1+\sqrt{3}} = \frac{-(3-\sqrt{3})(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{-(3-3+2\sqrt{3})}{3-1} = -\sqrt{3}$$

We can suppose that this equation represents some conic curve, though we do not understand the shape of the curve. we can understand the shape is hyperbola by following diagonalization

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} X_1 & X_2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = 1$$

Second example is $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Diagonalization

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2 - 1 = 0$$

$$\lambda - 2 = \pm 1$$

$$\lambda = 3, 1$$

$$2x_1 + x_2 = 3x_1$$

$$x_1 + 2x_2 = 3x_2$$

$$x_1 = x_2$$

$$\boldsymbol{e}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$2x_1 + x_2 = x_1$$

$$x_1 + 2x_2 = x_2$$

$$x_1 = -x_2$$

$$\boldsymbol{e}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Diagonalization matrix

$$\boldsymbol{P} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad \boldsymbol{P^{-1}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\boldsymbol{P^{-1}} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \boldsymbol{P}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} - \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} - \frac{2}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3+3}{2} & \frac{3-3}{2} \\ \frac{1-1}{2} & \frac{1+1}{2} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

Spectral decomposition

$$\lambda_1 \boldsymbol{e}_1 \boldsymbol{e}_1{}^T = 3 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = 3 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\lambda_2 \boldsymbol{e}_2 \boldsymbol{e}_2{}^T = 1 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \lambda_1 \boldsymbol{e}_1 \boldsymbol{e}_1{}^T + \lambda_2 \boldsymbol{e}_2 \boldsymbol{e}_2{}^T = 3 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} + 1 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= 3\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Using result of spectral decomposition, we make quadratic form

$$\boldsymbol{x}^T \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \boldsymbol{x} = \lambda_1 \boldsymbol{x}^T \boldsymbol{e}_1 {\boldsymbol{e}_1}^T \boldsymbol{x} + \lambda_2 \boldsymbol{x}^T \boldsymbol{e}_2 {\boldsymbol{e}_2}^T \boldsymbol{x}$$

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\lambda_1 \boldsymbol{x}^T \boldsymbol{e}_1 {\boldsymbol{e}_1}^T \boldsymbol{x} = 3(x_1 \quad x_2) \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3\left(\frac{x_1 + x_2}{2} \quad \frac{x_1 + x_2}{2}\right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= 3\left(\frac{x_1 + x_2}{2} \quad \frac{x_1 + x_2}{2}\right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{3}{2}({x_1}^2 + 2x_1x_2 + {x_1}^2) = 3\left(\frac{x_1 + x_2}{\sqrt{2}}\right)^2$$

$$\lambda_2 \boldsymbol{x}^T \boldsymbol{e}_2 {\boldsymbol{e}_2}^T \boldsymbol{x} = (x_1 \quad x_2) \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(\frac{x_1 - x_2}{2} \quad \frac{-x_1 + x_2}{2}\right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \frac{1}{2}({x_1}^2 - 2x_1x_2 + {x_1}^2) = \left(\frac{x_2 - x_1}{\sqrt{2}}\right)^2$$

$$\therefore \quad \lambda_1 \boldsymbol{x}^T \boldsymbol{e}_1 {\boldsymbol{e}_1}^T \boldsymbol{x} + \lambda_2 \boldsymbol{x}^T \boldsymbol{e}_2 {\boldsymbol{e}_2}^T \boldsymbol{x} = 3\left(\frac{x_1 + x_2}{\sqrt{2}}\right)^2 + \left(\frac{x_2 - x_1}{\sqrt{2}}\right)^2 \qquad \text{i}$$

$$(x_1 \quad x_2) \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (2x_1 + x_2 \quad x_1 + 2x_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2{x_1}^2 + 2x_1x_2 + 2{x_2}^2$$

$$= 3\left(\frac{x_1 + x_2}{\sqrt{2}}\right)^2 + \left(\frac{x_2 - x_1}{\sqrt{2}}\right)^2$$

$$x_1 + x_2 = X_1, \ x_2 - x_1 = X_2$$

$$(x_1 \quad x_2) \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{{X_1}^2}{\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2} + \frac{{X_2}^2}{\sqrt{2}^2} = 1$$

This a formula of ellipse. General form of ellipse is as follow.

$$\frac{{x_1}^2}{a^2} + \frac{{x_2}^2}{b^2} = 1$$

Ellipse is locus point where the absolute value of sum of the distances from two focuses is constant. When the focus points (P and Q) exist on the horizontal axis

$$\text{P:} \ (f \quad 0)$$

$$\text{Q:} \ (f \quad 0)$$

$$|L\text{Q}| + |L\text{P}| = 2a$$

$$|LQ|^2 + 2|LQ||LP| + |LP|^2 = 4a^2$$

$$|LP| = \sqrt{(x-f)^2 + (y-0)^2}$$

$$|LQ| = \sqrt{(x+f)^2 + (y-0)^2}$$

$$(x+f)^2 + y^2 + 2\sqrt{(x-f)^2 + y^2}\sqrt{(x+f)^2 + y^2} + (x-f)^2 + y^2 = 4a^2$$

$$2x^2 + 2f^2 + 2y^2 + 2\sqrt{(x^2 - f^2)^2 + 2x^2y^2 + 2f^2y^2 + y^4} = 4a^2$$

$$\sqrt{(x^2 - f^2)^2 + 2x^2y^2 + 2f^2y^2 + y^4} = 2a^2 - f^2 - x^2 - y^2$$

$$x^4 - 2x^2f^2 + f^4 + 2x^2y^2 + 2f^2y^2 + y^4$$
$$= 4a^4 - 4a^2f^2 - 4a^2x^2 - 4a^2y^2 + f^4 + 2f^2x^2 + 2f^2y^2 + x^4 + 2x^2y^2 + y^4$$

$$-2x^2f^2 = 4a^4 - 4a^2f^2 - 4a^2x^2 - 4a^2y^2 + 2f^2x^2$$

$$4a^2x^2 + 4a^2y^2 - 4f^2x^2 = 4a^4 - 4a^2f^2$$

$$(a^2 - f^2)x^2 + a^2y^2 = a^2(a^2 - f^2)$$

$$\frac{x^2}{a^2} + \frac{y^2}{(f^2 - a^2)} = 1$$

Let $(f^2 - a^2) = b^2$

$$f^2 = a^2 + b^2$$

$$f = \pm\sqrt{a^2 + b^2}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

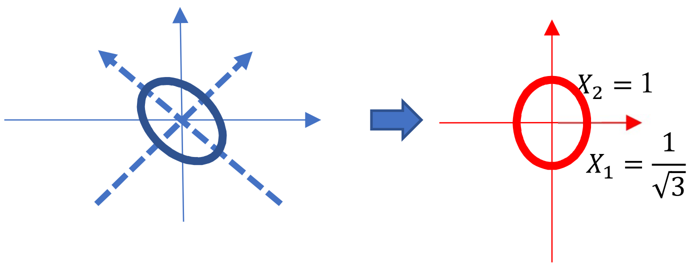

Fig. 57 Diagonalization of positive definite matrix

Shape of locus of the quadratic form is ellipse. When all eigenvalues are positive, the matrix is positive definite. Another definition of positive definite is "quadratic" form of the matrix is positive.

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\lambda_1 = 3 > 0, \qquad \lambda_2 = 1 > 0$$

$$(x_1 \quad x_2)\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (2x_1 + x_2 \quad x_1 + 2x_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2{x_1}^2 + 2x_1x_2 + 2{x_2}^2 == 3\left(\frac{x_1 + x_2}{\sqrt{2}}\right)^2 + \left(\frac{x_2 - x_1}{\sqrt{2}}\right)^2 > 0$$

Quadratic form of the matrix which is positive definite is hyperelliptic. In this case, we have only 2 variables. It may not so difficult to image the shape of the locus without using diagonalization as calculation technique. However, when variables increase more than 3, it is difficult to decompose formula to image the shape without diagonalization. Diagonalization of quadratic form is transformation of principle axis and transformed diagonalized quadratic form is canonical form of conic curves.

V-2-4-3. Spectral decomposition of quadratic form

Spectral decomposition of quadratic form is an application of diagonalization of quadratic form.

$$\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = \lambda_1\boldsymbol{x}^T\boldsymbol{e}_1\boldsymbol{e}_1{}^T\boldsymbol{x} + \lambda_2\boldsymbol{x}^T\boldsymbol{e}_2\boldsymbol{e}_2{}^T\boldsymbol{x} + \cdots \lambda_p\boldsymbol{x}^T\boldsymbol{e}_p\boldsymbol{e}_p{}^T\boldsymbol{x}$$

The formula $\boldsymbol{x}^T\boldsymbol{e}_i$ is linear form, and $\boldsymbol{x}^T\boldsymbol{e}_i\boldsymbol{e}_i{}^T\boldsymbol{x}$ is quadratic form as shown in following trans formation

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3\left(\frac{x_1 + x_2}{\sqrt{2}}\right)^2 + \left(\frac{x_2 - x_1}{\sqrt{2}}\right)^2$$

The formula of $\boldsymbol{x}^T\boldsymbol{e}_i\boldsymbol{e}_i{}^T\boldsymbol{x}$ is square of eigenvector. The meaning of spectral decomposition is transformation of quadratic form to sum of squares of eigenvectors.