V-2-5. Power of matrix

We consider A^m.

Supposing that $\boldsymbol{A}$ can be diagonalized.

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{\Lambda}$$

$$\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_p \end{pmatrix}$$

$\boldsymbol{\Lambda}$ is diagonal matrix

$$\boldsymbol{\Lambda}^m = \begin{pmatrix} {\lambda_1}^m & 0 & 0 & 0 \\ 0 & {\lambda_2}^m & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & {\lambda_p}^m \end{pmatrix}$$

Confirmation

$$\boldsymbol{\Lambda}^2 = \boldsymbol{\Lambda}\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_p \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_p \end{pmatrix} = \begin{pmatrix} {\lambda_1}^2 & 0 & 0 & 0 \\ 0 & {\lambda_2}^2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & {\lambda_p}^2 \end{pmatrix}$$

$$\boldsymbol{\Lambda}^3 = \boldsymbol{\Lambda}^2\boldsymbol{\Lambda} = \begin{pmatrix} {\lambda_1}^2 & 0 & 0 & 0 \\ 0 & {\lambda_2}^2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & {\lambda_p}^2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_p \end{pmatrix} = \begin{pmatrix} {\lambda_1}^3 & 0 & 0 & 0 \\ 0 & {\lambda_2}^3 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & {\lambda_p}^3 \end{pmatrix}$$

We raise both sides of following equation to the m^{th} power

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{\Lambda}$$

$$(\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P})^m = \boldsymbol{\Lambda}^m$$

$$\text{Left side} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} \cdots \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$$

$$\boldsymbol{P}\boldsymbol{P}^{-1} = \boldsymbol{I}$$

$$\text{Left side} = \boldsymbol{P}^{-1}\boldsymbol{A}^m\boldsymbol{P}$$

$$\therefore \boldsymbol{P}^{-1}\boldsymbol{A}^m\boldsymbol{P} = \boldsymbol{\Lambda}^m$$

Multiplicate $\boldsymbol{P}$ and $\boldsymbol{P}^{-1}$from left and right respectively to both side

$$\boldsymbol{P}\boldsymbol{P}^{-1}\boldsymbol{A}^m\boldsymbol{P}\boldsymbol{P}^{-1} = \boldsymbol{P}\boldsymbol{\Lambda}^m\boldsymbol{P}^{-1}$$

$$\boldsymbol{P}\boldsymbol{P}^{-1} = \boldsymbol{I}$$

$$\boldsymbol{A}^m = \boldsymbol{P}\boldsymbol{\Lambda}^m\boldsymbol{P}^{-1}$$

Formula 62

This means that we can express $\boldsymbol{A}^m$ as follow using spectral decomposition for symmetric matrix and quasi spectral decomposition for asymmetric matrix

Symmetric matrix

$$A^m = \lambda_1{}^m P_1 P_1{}^T + \lambda_2{}^m P_2 P_2{}^T + \cdots \lambda_p{}^m P_p P_p{}^T$$

P_i: eigen vector

$$\because\ P^{-1} = P^T$$

asymmetric matrix

$$A^m = \lambda_1{}^m P_1 P^{-1} + \lambda_2{}^m P_2 P^{-1} + \cdots \lambda_p{}^m P_p P^{-1}$$

$$P_1 = \begin{pmatrix} p_{11} & 0 & \cdots & 0 \\ p_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & 0 & \cdots & 0 \end{pmatrix},\quad P_2 = \begin{pmatrix} 0 & p_{12} & \cdots & 0 \\ 0 & p_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & p_{n2} & \cdots & 0 \end{pmatrix}, \cdots,\quad P_p = \begin{pmatrix} 0 & 0 & \cdots & p_{1n} \\ 0 & 0 & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_{nn} \end{pmatrix}$$

Example　(diagonalization of this example is calculated in V-2-2 Diagonalization)

$$A = \begin{pmatrix} 8 & 5 & 7 \\ -1 & 2 & -1 \\ -3 & -3 & -2 \end{pmatrix}$$

$$P = \begin{pmatrix} -1 & 3 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Diagonalization

$$P^{-1}AP = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 8 & 5 & 7 \\ -1 & 2 & -1 \\ -3 & -3 & -2 \end{pmatrix}\begin{pmatrix} -1 & 3 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad P_2 = \begin{pmatrix} 0 & 3 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix},\quad P_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P = P_1 + P_2 + P_3$$

$$P_1 P^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -2 & -1 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$P_2 P^{-1} = \begin{pmatrix} 0 & 3 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 3 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

$$P_3 P^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

$$A^2 = 3^2\begin{pmatrix} -1 & -2 & -1 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} + 4^2\begin{pmatrix} 3 & 3 & 3 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} + 1^2\begin{pmatrix} -1 & -1 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -9+48-1 & -18+48-1 & -9+48-2 \\ 9-16+0 & 18-16+0 & 9-16+0 \\ 0-16+1 & 0-16+1 & 0-16+2 \end{pmatrix}$$

$$= \begin{pmatrix} 38 & 29 & 37 \\ -7 & 2 & -7 \\ -15 & -15 & -14 \end{pmatrix}$$

$$\boldsymbol{A^3} = \boldsymbol{3^3} \begin{pmatrix} -1 & -2 & -1 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} + 4^3 \begin{pmatrix} 3 & 3 & 3 \\ -\ 1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} + 1^3 \begin{pmatrix} -1 & -1 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -27+192-1 & -54+192-1 & -27+192-2 \\ 27-64+0 & 54-64+0 & 27-64+0 \\ 0-64+1 & 0-64+1 & 0-64+2 \end{pmatrix}$$

$$= \begin{pmatrix} 164 & 137 & 163 \\ -37 & -10 & -37 \\ -63 & -63 & -62 \end{pmatrix}$$

Confirmation

$$\boldsymbol{A^2} = \begin{pmatrix} 8 & 5 & 7 \\ -1 & 2 & -1 \\ -3 & -3 & -2 \end{pmatrix} \begin{pmatrix} 8 & 5 & 7 \\ -1 & 2 & -1 \\ -3 & -3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 64-5-21 & 40+10-21 & 56-5-14 \\ -8-2+3 & -5+4+3 & -7-2+2 \\ -24+3+6 & -15-6+6 & -21+3+4 \end{pmatrix}$$

$$= \begin{pmatrix} 38 & 29 & 37 \\ -7 & 2 & -7 \\ -15 & -15 & -14 \end{pmatrix}$$

$$\boldsymbol{A^3} = \begin{pmatrix} 8 & 5 & 7 \\ -1 & 2 & -1 \\ -3 & -3 & -2 \end{pmatrix} \begin{pmatrix} 8 & 5 & 7 \\ -1 & 2 & -1 \\ -3 & -3 & -2 \end{pmatrix} \begin{pmatrix} 8 & 5 & 7 \\ -1 & 2 & -1 \\ -3 & -3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 38 & 29 & 37 \\ -7 & 2 & -7 \\ -15 & -15 & -14 \end{pmatrix} \begin{pmatrix} 8 & 5 & 7 \\ -1 & 2 & -1 \\ -3 & -3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 38\times 8-29-37\times 3 & 38\times 5+29\times 2-37\times 3 & 38\times 7-29\times 1-37\times 2 \\ -7\times 8-2\times 1+7\times 3 & -15\times 5+2\times 2+7\times 3 & -15\times 7-2\times 1+7\times 2 \\ -15\times 8+15\times 1+14\times 3 & -15\times 5-15\times 2+14\times 3 & -15\times 7+15\times 1+14\times 2 \end{pmatrix}$$

$$\begin{pmatrix} 164 & 137 & 163 \\ -37 & -10 & -37 \\ -63 & -63 & -62 \end{pmatrix}$$