V-3. Utilization of variance-covariance matrix

V-3-1. Variance-covariance matrix

V-3-1-1 Calculation of variance and covariance

When we get data set, we make data sheet at first, and then we calculate fundamental statistic values such as average, maximum and minimum value and confirm the distribution of data. In addition to this, it is better to check the relation of observed characteristics, because estimated parameters are instable when explanatory variables have correlation each other (multicollinearity).

Analysts use spreadsheet software such as Excel, though they have their own form of data sheet. Following is a typical form of data sheet.

Table 41. An example of data

	Characteristics			
Sample no.	A	B	$\cdots$	P
1	d_{11}	d_{12}	$\cdots$	d_{1p}
2	d_{21}	d_{22}	$\cdots$	d_{2p}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
n	d_{n1}	d_{n2}	$\cdots$	d_{np}
Average	$\bar{d}_1 = \frac{1}{n}\sum_{i=1}^{n} d_{i1}$	$\bar{d}_2 = \frac{1}{n}\sum_{i=1}^{n} d_{i2}$	$\cdots$	$\bar{d}_p = \frac{1}{n}\sum_{i=1}^{n} d_{ip}$

Then the data are transformed to the difference from the average as follow

$$c_{ij} = d_{ij} - \bar{d}_j$$

Table 42. standardization of data by distance from the mean.

	Characteristics			
Sample no.	A	B	$\cdots$	P
1	c_{11}	c_{12}	$\cdots$	c_{1p}
2	c_{21}	c_{22}	$\cdots$	c_{2p}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
n	c_{n1}	c_{n2}	$\cdots$	c_{np}

We can make variance covariance matrix from table 42.

$$\boldsymbol{c}^T = (\boldsymbol{c_1} \quad \boldsymbol{c_2} \quad \cdots \quad \boldsymbol{c_p}) = \begin{pmatrix} c_{11} & c_{21} & \cdots & c_{p1} \\ c_{12} & c_{22} & \cdots & c_{p2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1n} & c_{2n} & \cdots & c_{pn} \end{pmatrix}$$

$$\boldsymbol{c} = \begin{pmatrix} \boldsymbol{c_1} \\ \boldsymbol{c_2} \\ \vdots \\ \boldsymbol{c_p} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{pmatrix}$$

$$\boldsymbol{cc}^T = \begin{pmatrix} \boldsymbol{c_1} \\ \boldsymbol{c_2} \\ \vdots \\ \boldsymbol{c_p} \end{pmatrix} (\boldsymbol{c_1} \quad \boldsymbol{c_2} \quad \cdots \quad \boldsymbol{c_p}) = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{pmatrix} \begin{pmatrix} c_{11} & c_{21} & \cdots & c_{p1} \\ c_{12} & c_{22} & \cdots & c_{p2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1n} & c_{2n} & \cdots & c_{pn} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=1}^{n} c_{1k}c_{1k} & \sum_{k=1}^{n} c_{1k}c_{2k} & \cdots & \sum_{k=1}^{n} c_{1k}c_{pk} \\ \sum_{k=1}^{n} c_{2k}c_{1k} & \sum_{k=1}^{n} c_{2k}c_{2k} & \cdots & \sum_{k=1}^{n} c_{2k}c_{pk} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} c_{pk}c_{1k} & \sum_{k=1}^{n} c_{pk}c_{2k} & \cdots & \sum_{k=1}^{n} c_{pk}c_{pk} \end{pmatrix}$$

Factors of the matrix are sum of square. Variance and covariance are obtainable dividing the factors by degree of freedom. When our purpose is to obtain secondary moment of around expectation values of mean of parent population, the degree of freedom should be $n-1$ or $n-2$. However, we are discussing relation of factors in sample population. So we use n as degree of freedom.

$$\boldsymbol{E}(\boldsymbol{cc}^T) = \frac{1}{n} \begin{pmatrix} \sum_{k=1}^{n} c_{1k}c_{1k} & \sum_{k=1}^{n} c_{1k}c_{2k} & \cdots & \sum_{k=1}^{n} c_{1k}c_{pk} \\ \sum_{k=1}^{n} c_{2k}c_{1k} & \sum_{k=1}^{n} c_{2k}c_{2k} & \cdots & \sum_{k=1}^{n} c_{2k}c_{pk} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} c_{pk}c_{1k} & \sum_{k=1}^{n} c_{pk}c_{2k} & \cdots & \sum_{k=1}^{n} c_{pk}c_{pk} \end{pmatrix}$$

$\boldsymbol{E}(\boldsymbol{cc}^T)$ is an expected matrix when we randomly pick up data from sample population. Commonly, this matrix is denoted as $\boldsymbol{\Sigma}$, and the factors in the matrix are expressed as σ_{ij}.

$$\boldsymbol{\Sigma} = \boldsymbol{E}(\boldsymbol{cc}^T) = \frac{1}{n} \begin{pmatrix} \sum_{k=1}^{n} c_{1k}c_{1k} & \sum_{k=1}^{n} c_{1k}c_{2k} & \cdots & \sum_{k=1}^{n} c_{1k}c_{pk} \\ \sum_{k=1}^{n} c_{2k}c_{1k} & \sum_{k=1}^{n} c_{2k}c_{2k} & \cdots & \sum_{k=1}^{n} c_{2k}c_{pk} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} c_{pk}c_{1k} & \sum_{k=1}^{n} c_{pk}c_{2k} & \cdots & \sum_{k=1}^{n} c_{pk}c_{pk} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{p2} \end{pmatrix}$$

Formula 67

This is symmetric matrix because $\sigma_{ij} = \sigma_{ji}$. Diagonal factor σ_{ii} is variance and it is generally expressed as $\sigma_i{}^2$. However, σ_{ii} is a notation for specifying calculation process. From variance covariance matrix, we can make correlation matrix. Correlation coefficient ρ_{ij} is as follow.

$$\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}}\sqrt{\sigma_{jj}}}$$

Correlation matrix $\boldsymbol{\rho}$ is

$$\boldsymbol{\rho} = \begin{pmatrix} \frac{\sigma_{11}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{11}}} & \frac{\sigma_{12}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}} & \cdots & \frac{\sigma_{1p}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{pp}}} \\ \frac{\sigma_{21}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{11}}} & \frac{\sigma_{22}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{22}}} & \cdots & \frac{\sigma_{2p}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{pp}}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sigma_{p1}}{\sqrt{\sigma_{pp}}\sqrt{\sigma_{11}}} & \frac{\sigma_{p2}}{\sqrt{\sigma_{pp}}\sqrt{\sigma_{22}}} & \cdots & \frac{\sigma_{pp}}{\sqrt{\sigma_{pp}}\sqrt{\sigma_{pp}}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1p} \\ \rho_{21} & 1 & \cdots & \rho_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{p1} & \rho_{p2} & \cdots & 1 \end{pmatrix}$$

Formula 68

When we denote Variance matrix as $\boldsymbol{V}$

$$\boldsymbol{V} = \begin{pmatrix} \sigma_{11} & 0 & \cdots & 0 \\ 0 & \sigma_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{pp} \end{pmatrix}$$

Using power method of matrix

$$\boldsymbol{V}^{\frac{1}{2}} = \begin{pmatrix} \sqrt{\sigma_{11}} & 0 & \cdots & 0 \\ 0 & \sqrt{\sigma_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\sigma_{pp}} \end{pmatrix}$$

The relation among $\boldsymbol{\Sigma}$, $\boldsymbol{\rho}$ and $\boldsymbol{V}$ is as follow.

$$\boldsymbol{V}^{\frac{1}{2}}\boldsymbol{\rho}\boldsymbol{V}^{\frac{1}{2}} = \boldsymbol{\Sigma}$$

$$\boldsymbol{V}^{-\frac{1}{2}}\boldsymbol{\Sigma}\boldsymbol{V}^{-\frac{1}{2}} = \boldsymbol{\rho}$$

Formula 69

Confirmation

$$\begin{pmatrix} \sqrt{\sigma_{11}} & 0 & \cdots & 0 \\ 0 & \sqrt{\sigma_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\sigma_{pp}} \end{pmatrix}^{-1} \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{pmatrix} \begin{pmatrix} \sqrt{\sigma_{11}} & 0 & \cdots & 0 \\ 0 & \sqrt{\sigma_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\sigma_{pp}} \end{pmatrix}^{-1}$$

$$\begin{pmatrix} \frac{1}{\sqrt{\sigma_{11}}} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sqrt{\sigma_{22}}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sqrt{\sigma_{pp}}} \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sigma_{11}}{\sqrt{\sigma_{11}}} & \frac{\sigma_{12}}{\sqrt{\sigma_{11}}} & \cdots & \frac{\sigma_{1p}}{\sqrt{\sigma_{11}}} \\ \frac{\sigma_{21}}{\sqrt{\sigma_{22}}} & \frac{\sigma_{22}}{\sqrt{\sigma_{22}}} & \cdots & \frac{\sigma_{2p}}{\sqrt{\sigma_{22}}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sigma_{p1}}{\sqrt{\sigma_{pp}}} & \frac{\sigma_{p2}}{\sqrt{\sigma_{pp}}} & \cdots & \frac{\sigma_{pp}}{\sqrt{\sigma_{pp}}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\sigma_{11}}{\sqrt{\sigma_{11}}} & \frac{\sigma_{12}}{\sqrt{\sigma_{11}}} & \cdots & \frac{\sigma_{1p}}{\sqrt{\sigma_{11}}} \\ \frac{\sigma_{21}}{\sqrt{\sigma_{22}}} & \frac{\sigma_{22}}{\sqrt{\sigma_{22}}} & \cdots & \frac{\sigma_{2p}}{\sqrt{\sigma_{22}}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sigma_{p1}}{\sqrt{\sigma_{pp}}} & \frac{\sigma_{p2}}{\sqrt{\sigma_{pp}}} & \cdots & \frac{\sigma_{pp}}{\sqrt{\sigma_{pp}}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\sigma_{11}}} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sqrt{\sigma_{22}}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sqrt{\sigma_{pp}}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sigma_{11}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{11}}} & \frac{\sigma_{12}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}} & \cdots & \frac{\sigma_{1p}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{pp}}} \\ \frac{\sigma_{21}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{11}}} & \frac{\sigma_{22}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{22}}} & \cdots & \frac{\sigma_{2p}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{pp}}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sigma_{p1}}{\sqrt{\sigma_{pp}}\sqrt{\sigma_{11}}} & \frac{\sigma_{p2}}{\sqrt{\sigma_{pp}}\sqrt{\sigma_{22}}} & \cdots & \frac{\sigma_{pp}}{\sqrt{\sigma_{pp}}\sqrt{\sigma_{pp}}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1p} \\ \rho_{21} & 1 & \cdots & \rho_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{p1} & \rho_{p2} & \cdots & 1 \end{pmatrix} = \boldsymbol{\rho}$$

$$\boldsymbol{V}^{-\frac{1}{2}}\boldsymbol{\Sigma}\boldsymbol{V}^{-\frac{1}{2}} = \boldsymbol{\rho}$$