V-3-2. Structure of variance covariance matrix

The relation among variance covariance matrix, Variance matrix and correlation matrix is as follow.

$$\boldsymbol{V}^{\frac{1}{2}}\boldsymbol{\rho}\boldsymbol{V}^{\frac{1}{2}} = \boldsymbol{\Sigma}$$

$$\boldsymbol{V}^{-\frac{1}{2}}\boldsymbol{\Sigma}\boldsymbol{V}^{-\frac{1}{2}} = \boldsymbol{\rho}$$

It is difficult to express the relation in multidimensional space not only because of limitation in space reasoning capacity but also because of complexity of calculation. However, it is not so difficult to express the relation in two-dimensional flat.

Correlation matrix in two-dimension is as follow

$$\boldsymbol{\rho} = \begin{pmatrix} 1 & \delta_{12} \\ \delta_{21} & 1 \end{pmatrix} = \begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix}$$

$$\because \sqrt{\sigma_{11}}\sqrt{\sigma_{22}}\cos\theta = \sigma_{12}$$

$$\delta_{12} = \frac{\sigma_{12}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}} = \cos\theta$$

θ: angle between vector of factor 1 and factor 2

Setting up eigenvalue and eigenvector

$$\begin{vmatrix} 1-\lambda & \cos\theta \\ \cos\theta & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 - \cos^2\theta = 0$$

$$\lambda = 1 \pm \cos\theta$$

$$\begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (1+\cos\theta)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x_1 + x_2\cos\theta = x_1 + x_1\cos\theta$$

$$x_2 + x_1\cos\theta = x_2 + x_2\cos\theta$$

$$x_1 = x_2$$

$$\boldsymbol{e_1} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (1-\cos\theta)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x_1 + x_2\cos\theta = x_1 - x_1\cos\theta$$

$$x_2 + x_1\cos\theta = x_2 - x_2\cos\theta$$

$$x_1 = -x_2$$

$$\boldsymbol{e_2} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\boldsymbol{\rho} = (1 + \cos\theta)\boldsymbol{e_1}\boldsymbol{e_1}^{\boldsymbol{T}} + (1 - \cos\theta)\boldsymbol{e_2}\boldsymbol{e_2}^{\boldsymbol{T}}$$

$$\boldsymbol{\rho}^{\frac{1}{2}} = \sqrt{1 + \cos\theta} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} + \sqrt{1 - \cos\theta} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\boldsymbol{\rho}^{\frac{1}{2}} = \sqrt{1 + \cos\theta} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} + \sqrt{1 - \cos\theta} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\boldsymbol{\rho}^{\frac{1}{2}} = \sqrt{1 + \cos\theta} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \sqrt{1 - \cos\theta} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\boldsymbol{\rho}^{\frac{1}{2}} = \frac{1}{2} \begin{pmatrix} \sqrt{1 + \cos\theta} + \sqrt{1 - \cos\theta} & \sqrt{1 + \cos\theta} - \sqrt{1 - \cos\theta} \\ \sqrt{1 + \cos\theta} - \sqrt{1 - \cos\theta} & \sqrt{1 + \cos\theta} + \sqrt{1 - \cos\theta} \end{pmatrix}$$

$$\boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{\rho}\boldsymbol{x} = \boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{\rho}^{\frac{1}{2}}\boldsymbol{\rho}^{\frac{1}{2}}\boldsymbol{x}$$

$$\boldsymbol{x}^{T}\boldsymbol{\rho}^{\frac{1}{2}} = \left(\boldsymbol{\rho}^{\frac{1}{2}}\boldsymbol{x}\right)^{T}$$

$$\boldsymbol{\rho}^{\frac{1}{2}}\boldsymbol{x} = \frac{1}{2} \begin{pmatrix} \sqrt{1 + \cos\theta} + \sqrt{1 - \cos\theta} & \sqrt{1 + \cos\theta} - \sqrt{1 - \cos\theta} \\ \sqrt{1 + \cos\theta} - \sqrt{1 - \cos\theta} & \sqrt{1 + \cos\theta} + \sqrt{1 - \cos\theta} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} \left(\sqrt{1 + \cos\theta} + \sqrt{1 - \cos\theta}\right)x_1 + \left(\sqrt{1 + \cos\theta} - \sqrt{1 - \cos\theta}\right)x_2 \\ \left(\sqrt{1 + \cos\theta} - \sqrt{1 - \cos\theta}\right)x_1 + \left(\sqrt{1 + \cos\theta} + \sqrt{1 - \cos\theta}\right)x_2 \end{pmatrix}$$

We consider a circle of radius 1 in figure 60. Vector $\boldsymbol{A}: \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a vector on the horizontal axis. $\boldsymbol{\acute{A}} = \boldsymbol{\rho}^{\frac{1}{2}}\boldsymbol{A}$ is transformed vector from $\boldsymbol{A}$ by $\boldsymbol{\rho}^{\frac{1}{2}}$.

$$\boldsymbol{\acute{A}} = \frac{1}{2} \begin{pmatrix} \sqrt{1 + \cos\theta} + \sqrt{1 - \cos\theta} & \sqrt{1 + \cos\theta} - \sqrt{1 - \cos\theta} \\ \sqrt{1 + \cos\theta} - \sqrt{1 - \cos\theta} & \sqrt{1 + \cos\theta} + \sqrt{1 - \cos\theta} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{1 + \cos\theta} + \sqrt{1 - \cos\theta}}{2} \\ \left(\frac{\sqrt{1 + \cos\theta} - \sqrt{1 - \cos\theta}}{2}\right) \end{pmatrix}$$

$$\left|\acute{\boldsymbol{A}}\right| = \sqrt{\left(\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{2}\right)^{\boldsymbol{2}} + \left(\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{2}\right)^{2}} = 1$$

When $\boldsymbol{B} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, and\ \acute{\boldsymbol{B}} = \boldsymbol{\rho}^{\frac{1}{2}}\boldsymbol{B}$

$$\acute{\boldsymbol{B}} = \frac{1}{2}\begin{pmatrix} \sqrt{1+\cos\theta}+\sqrt{1-\cos\theta} & \sqrt{1+\cos\theta}-\sqrt{1-\cos\theta} \\ \sqrt{1+\cos\theta}-\sqrt{1-\cos\theta} & \sqrt{1+\cos\theta}+\sqrt{1-\cos\theta} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{1+\cos\theta}}{\sqrt{2}} \\ \frac{\sqrt{1+\cos\theta}}{\sqrt{2}} \end{pmatrix}$$

$$\left|\acute{\boldsymbol{B}}\right| = \sqrt{\left(\frac{\sqrt{1+\cos\theta}}{\sqrt{2}}\right)^{\boldsymbol{2}} + \left(\frac{\sqrt{1+\cos\theta}}{\sqrt{2}}\right)^{2}} = \sqrt{1+\cos\theta}$$

When $\boldsymbol{C} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, and\ \acute{\boldsymbol{C}} = \boldsymbol{\rho}^{\frac{1}{2}}\boldsymbol{C}$

$$\boldsymbol{C} = \frac{1}{2}\begin{pmatrix} \sqrt{1+\cos\theta}+\sqrt{1-\cos\theta} & \sqrt{1+\cos\theta}-\sqrt{1-\cos\theta} \\ \sqrt{1+\cos\theta}-\sqrt{1-\cos\theta} & \sqrt{1+\cos\theta}+\sqrt{1-\cos\theta} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{2} \\ \left(\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{2}\right) \end{pmatrix}$$

$$\left|\acute{\boldsymbol{C}}\right| = \sqrt{\left(\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{2}\right)^{\boldsymbol{2}} + \left(\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{2}\right)^{2}} = 1$$

When $\boldsymbol{D} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, and\ \acute{\boldsymbol{D}} = \boldsymbol{\rho}^{\frac{1}{2}}\boldsymbol{D}$

$$\acute{\boldsymbol{D}} = \frac{1}{2}\begin{pmatrix} \sqrt{1+\cos\theta}+\sqrt{1-\cos\theta} & \sqrt{1+\cos\theta}-\sqrt{1-\cos\theta} \\ \sqrt{1+\cos\theta}-\sqrt{1-\cos\theta} & \sqrt{1+\cos\theta}+\sqrt{1-\cos\theta} \end{pmatrix}\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\sqrt{1-\cos\theta}}{\sqrt{2}} \\ \frac{\sqrt{1-\cos\theta}}{\sqrt{2}} \end{pmatrix}$$

$$|\acute{\boldsymbol{D}}| = \sqrt{\left(\frac{\sqrt{1-\cos\theta}}{\sqrt{2}}\right)^{\mathbf{2}} + \left(-\frac{\sqrt{1-\cos\theta}}{\sqrt{2}}\right)^{2}} = \sqrt{1-\cos\theta}$$

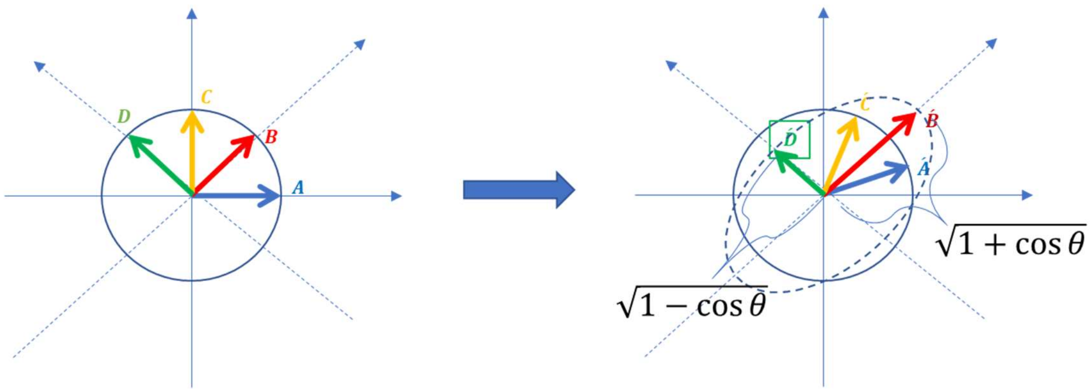

Fig 60. Transformation by $\boldsymbol{\rho}^{\frac{1}{2}}\boldsymbol{x}$

From figure 60, we can understand that function of $\boldsymbol{\rho}^{\frac{1}{2}}$ expand the vector $\sqrt{1+\cos\theta}$ times to direction a eigenvector and diminish the vector $\sqrt{1-\cos\theta}$ to right angle. We can understand easily the function in case of two-dimensional locus, though it is impossible to draw the function of $\boldsymbol{\rho}^{\frac{1}{2}}$ in multi-dimensional space. However, when we pick up the relation between different two-dimensional spaces, we can express the relation as in figure 60.

About the variance matrix, $\mathbf{V}^{-\frac{1}{2}}$, **we can express the function simply as follow.**

$$\boldsymbol{X} = \mathbf{V}^{-\frac{1}{2}}\grave{\mathbf{X}} = \begin{pmatrix} \frac{1}{\sqrt{\sigma_{11}}} & 0 \\ 0 & \frac{1}{\sqrt{\sigma_{22}}} \end{pmatrix} \begin{pmatrix} \acute{X}_1 \\ \acute{X}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\sigma_{11}}}\acute{X}_1 \\ \frac{1}{\sqrt{\sigma_{22}}}\acute{X}_2 \end{pmatrix}$$

Using $\boldsymbol{\rho}^{\frac{1}{2}}$ and $\mathbf{V}^{-\frac{1}{2}}$, we can plot original data to the transformed space which are composed from orthogonal factors and standardized by variance.