VI-1-2. Multicollinearity and partial correlation analysis.
VI-1-2-1. What is partial correlation analysis

Multicollinearity complicate interpretation of multiple linear regression. There are two types of mechanism of multicollinearity. One is overlapping of similar explanatory variables. The other mechanism is spurious correlation. Including both body height and sitting height in explanatory variables without any relativizing is overlapping, because sitting height is an element of body height. As shown in example in paragraph VI-1-1, we can observe correlation between sprint speed and body weight. However, we cannot conclude that heavy people move quickly from this fact. This is a typical spurious correlation. Sprint speed is strongly related with body size and body weight is strongly correlated with body size, particularly when the dataset is obtained from the population which includes all ages including children and adults. When we implement an experiment to detect the relation between sprint speed and body weight getting samples from a population in which all subject have same body height, we can observe inverse correlation. Multicollinearity is made by combination of overlapping and spurious correlation. Countermeasure to multicollinearity differs depending on strength of spurious correlation. When multicollinearity happens only overlapping of explanatory variables, we can solve multicollinearity by selection a variable from variables which have multicollinearity. When we kick out a variable from explanatory variables in the case when spurious correlation is large, we may miss a chance to get new findings, because, possibly, there exists important inverse correlation among variables. Generally, in the field of natural science, it is not so difficult to find spurious correlation, because causal connection is relatively simple. However, causal connection is complicated in the field of social science and psychology. We need method to detect spurious correlation statistically. Partial correlation analysis is method to detect spurious correlation.

VI-1-2-2. Mechanism of spurious correlation

Real world is composed from networks of relation. Actual mechanism of spurious correlation is more complicated, though most simple relation among three variables can be expressed as figure 65. Vector $\overrightarrow{OW}$ and $\overrightarrow{OF}$ is exist on W-O-F plane. Vector $\overrightarrow{OM}$ is orthogonal to the plane. Angle between vector $\overrightarrow{OW}$ and $\overrightarrow{OF}$ is θ. When we incline $\overrightarrow{OW}$ and $\overrightarrow{OF}$ to $\overrightarrow{OM}$ on M-W-O plane and M-F-O plane, we get Vector $\overrightarrow{OW'}$ and $\overrightarrow{OF'}$.

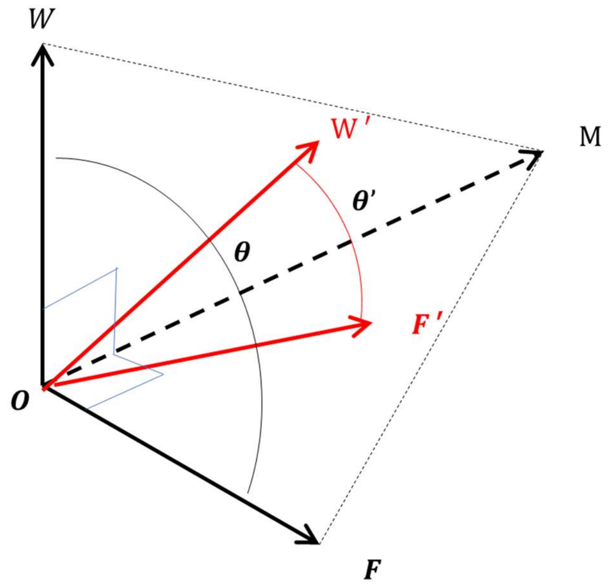

Fig. 65. Geometrical illustration of spurious correlation

The angle between vector $\overrightarrow{OW'}$ and $\overrightarrow{OF'}$ is θ'. Angle θ' is smaller than angle θ.

$$r_{WF} = \cos\theta$$

$$r_{W'F'} = \cos\theta'$$

$$\theta > \theta'$$

$$\cos\theta < \cos\theta'$$

$$-1 \le r_{WF} < r_{W'F'} \le 1$$

When the space is n-dimension, the vector expressed as follows

$$\boldsymbol{M} = \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}, \quad \boldsymbol{W} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}, \quad \boldsymbol{F} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix},$$

Assuming correlation among variables, predict value of $\widehat{m_i}$ is expressed as follow.

$$\widehat{m_i} = a_1 w_i + a_2 f_i$$

$$m_i = \widehat{m_i} + e_1$$

From this we can express predict vector of $\widehat{\boldsymbol{M}}$ as linear combination of vector $\boldsymbol{W}$and $\boldsymbol{F}$.

$$\widehat{\boldsymbol{M}} = a_1\boldsymbol{W} + a_2\,\boldsymbol{F}$$

$$\boldsymbol{M} = \widehat{\boldsymbol{M}} + \boldsymbol{E} = a_1\boldsymbol{W} + a_2\,\boldsymbol{F} + \boldsymbol{E}$$

We consider

$\boldsymbol{M}$: data of body size such as body height expressed by deviation from average.

$\boldsymbol{W}$: data of body weight expressed by deviation from average.

$\boldsymbol{F}$: data of sprint speed expressed by deviation from average.

When we get data from people whose body height is the same, $\boldsymbol{M}$ =0, and data distribute on W-O-F plane. So, the correlation coefficient is $r_{WF} = \cos\theta$. When we get data from population whose body height is not control, the population include various body height, and $\boldsymbol{M} \neq 0$, the allow head of vector $\boldsymbol{W}$ and $\boldsymbol{F}$ move to $\mathbf{W}'$ and $\mathbf{F}'$, and predictive vector $\widehat{\boldsymbol{M}}$ is linear combination of vector $\mathbf{W}'$and $\mathbf{F}$, and $\widehat{\boldsymbol{M}}$ exist on W'-O-F' plane.

As in figure 65,

$$\theta > \theta'$$

$$\cos\theta < \cos\theta'$$

$$-1 \leq r_{WF} < r_{W'F'} \leq 1$$

This is the mechanism of spurious correlation.

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VI-1-2-3. Operation of partial correlation analysis

At first, we consider the case there are only three variables

$$\boldsymbol{x} = (x_{1,\cdots}x_n)$$

$$\boldsymbol{y} = (y_{1,\cdots}y_n)$$

$$\boldsymbol{z} = (z_{1,\cdots}z_n)$$

$$x_i = d_{xi} - \overline{d_x}$$

$$y_i = d_{yi} - \overline{d_y}$$

$$z_i = d_{zi} - \overline{d_z}$$

d_{xi}: ith data of variable x, $\overline{d_x}$: average of variable x. $\overline{d_x} = \frac{1}{n}\sum_{i=1}^{n} d_{xi}$

d_{yi}: ith data of variable y, $\overline{d_y}$: average of variable y. $\overline{d_y} = \frac{1}{n}\sum_{i=1}^{n} d_{yi}$

d_{zi}: ith data of variable z, $\overline{d_z}$: average of variable z. $\overline{d_z} = \frac{1}{n}\sum_{i=1}^{n} d_{zi}$

We can make variance covariance matrix and correlation matrix

$$\boldsymbol{S} = \begin{pmatrix} \sum_{i=1}^{n} x_i{}^2 & \sum_{i=1}^{n} x_i\, y_i & \sum_{i=1}^{n} x_i\, z_i \\ \sum_{i=1}^{n} y_i x_i & \sum_{i=1}^{n} y_i{}^2 & \sum_{i=1}^{n} y_i z_i \\ \sum_{i=1}^{n} z_i x_i & \sum_{i=1}^{n} z_i y_i & \sum_{i=1}^{n} y_i{}^2 \end{pmatrix} = \begin{pmatrix} SS_{xx} & SS_{xy} & SS_{xz} \\ SS_{yx} & SS_{yy} & SS_{yz} \\ SS_{zx} & SS_{zy} & SS_{yy} \end{pmatrix}$$

$$\boldsymbol{R}=\begin{pmatrix}\frac{SS_{xx}}{\sqrt{SS_{xx}}\sqrt{SS_{xx}}} & \frac{SS_{xy}}{\sqrt{SS_{xx}}\sqrt{SS_{yy}}} & \frac{SS_{xz}}{\sqrt{SS_{xx}}\sqrt{SS_{zz}}}\\ \frac{SS_{yx}}{\sqrt{SS_{yy}}\sqrt{SS_{xx}}} & \frac{SS_{yy}}{\sqrt{SS_{yy}}\sqrt{SS_{yy}}} & \frac{SS_{yz}}{\sqrt{SS_{yy}}\sqrt{SS_{zz}}}\\ \frac{SS_{zx}}{\sqrt{SS_{zz}}\sqrt{SS_{xx}}} & \frac{SS_{zy}}{\sqrt{SS_{zz}}\sqrt{SS_{yy}}} & \frac{SS_{zz}}{\sqrt{SS_{zz}}\sqrt{SS_{zz}}}\end{pmatrix}=\begin{pmatrix}1 & r_{xy} & r_{xz}\\ r_{yx} & 1 & r_{yz}\\ r_{zx} & r_{zy} & 1\end{pmatrix}$$

$$|\boldsymbol{R}|=1+2\frac{SS_{xy}SS_{yz}SS_{zx}}{SS_{xx}SS_{yy}SS_{zz}}-\frac{SS_{yy}{SS_{zx}}^2+SS_{zz}{SS_{xy}}^2+SS_{xx}{SS_{yz}}^2}{SS_{xx}SS_{yy}SS_{zz}}$$

$$=\frac{SS_{xx}SS_{yy}SS_{zz}+2SS_{xy}SS_{yz}SS_{zx}-SS_{yy}{SS_{zx}}^2-SS_{zz}{SS_{xy}}^2-SS_{xx}{SS_{yz}}^2}{SS_{xx}SS_{yy}SS_{zz}}$$

$$\boldsymbol{R}^{-1}=\frac{1}{|\boldsymbol{R}|}\widetilde{\boldsymbol{R}}=\frac{1}{|\boldsymbol{R}|}\begin{pmatrix}1-{r_{yz}}^2 & r_{yz}r_{zx}-r_{xy} & r_{xy}r_{yz}-r_{zx}\\ r_{yz}r_{zx}-r_{xy} & 1-{r_{zx}}^2 & r_{xy}r_{zx}-r_{yz}\\ r_{xy}r_{yz}-r_{xz} & r_{xy}r_{zx}-r_{yz} & 1-{r_{xy}}^2\end{pmatrix}=\begin{pmatrix}r^{xx} & r^{xy} & r^{xz}\\ r^{yx} & r^{yy} & r^{yz}\\ r^{zx} & r^{yz} & r^{zz}\end{pmatrix}$$

$$=\frac{1}{|\boldsymbol{R}|}\begin{pmatrix}1-\frac{{SS_{yz}}^2}{SS_{yy}SS_{zz}} & \frac{SS_{yz}SS_{zx}-SS_{xy}SS_{zz}}{\sqrt{SS_{yy}}\sqrt{SS_{xx}}SS_{zz}} & \frac{SS_{xy}SS_{yz}-SS_{xz}SS_{yy}}{\sqrt{SS_{xx}}\sqrt{SS_{zz}}SS_{yy}}\\ \frac{SS_{yz}SS_{zx}-SS_{xy}SS_{zz}}{\sqrt{SS_{yy}}\sqrt{SS_{xx}}SS_{zz}} & 1-\frac{{SS_{zx}}^2}{SS_{zz}SS_{xx}} & \frac{SS_{xy}SS_{zx}-SS_{yz}SS_{xx}}{\sqrt{SS_{yy}}\sqrt{SS_{zz}}SS_{xx}}\\ \frac{SS_{xy}SS_{yz}-SS_{xz}SS_{yy}}{\sqrt{SS_{xx}}\sqrt{SS_{zz}}SS_{yy}} & \frac{SS_{xy}SS_{zx}-SS_{yz}SS_{xx}}{\sqrt{SS_{yy}}\sqrt{SS_{zz}}SS_{xx}} & 1-\frac{{SS_{xy}}^2}{SS_{xx}SS_{yy}}\end{pmatrix}$$

$$=\frac{1}{|\boldsymbol{R}|}\begin{pmatrix}\frac{SS_{yy}SS_{zz}-{SS_{yz}}^2}{SS_{yy}SS_{zz}} & \frac{SS_{yz}SS_{zx}-SS_{xy}SS_{zz}}{\sqrt{SS_{yy}}\sqrt{SS_{xx}}SS_{zz}} & \frac{SS_{xy}SS_{yz}-SS_{xz}SS_{yy}}{\sqrt{SS_{xx}}\sqrt{SS_{zz}}SS_{yy}}\\ \frac{SS_{yz}SS_{zx}-SS_{xy}SS_{zz}}{\sqrt{SS_{yy}}\sqrt{SS_{xx}}SS_{zz}} & \frac{SS_{zz}SS_{xx}-{SS_{zx}}^2}{SS_{zz}SS_{xx}} & \frac{SS_{xy}SS_{zx}-SS_{yz}SS_{xx}}{\sqrt{SS_{yy}}\sqrt{SS_{zz}}SS_{xx}}\\ \frac{SS_{xy}SS_{yz}-SS_{xz}SS_{yy}}{\sqrt{SS_{xx}}\sqrt{SS_{zz}}SS_{yy}} & \frac{SS_{xy}SS_{zx}-SS_{yz}SS_{xx}}{\sqrt{SS_{yy}}\sqrt{SS_{zz}}SS_{xx}} & \frac{SS_{xx}SS_{yy}-{SS_{xy}}^2}{SS_{xx}SS_{yy}}\end{pmatrix}$$

Meaning of simple linear regression is explanation of a variable by another explanatory variable. The solution of simple linear regression is expressed as follow

$$x=\alpha_{x/z}z+e=\frac{SS_{zx}}{SS_{zz}}z+e$$

$$y=\alpha_{y/z}z+e=\frac{SS_{yz}}{SS_{zz}}z+e$$

$\alpha_{x/z}$: regression coefficient in smple linear regression between x and z, $\alpha_{x/z}=\frac{SS_{zx}}{SS_{zz}}$

$\alpha_{x/y}$: regression coefficient in smple linear regression between y and z, $\alpha_{x/z} = \frac{SS_{zx}}{SS_{zz}}$

In the case when we consider variables more then 4, $(x_1 \quad \cdots \quad x_p), p \geq 4$

$$x_1 = \alpha_{x_1/x_3}x_3 + \cdots + \alpha_{x_1/x_p}x_p + e$$

$$x_2 = \alpha_{x_2/x_3}x_3 + \cdots + \alpha_{x_2/x_p}x_p + e$$

$$\vdots$$

In these equations, we consider that e is residual which cannot be explained by other variables other than x_1 and x_2. However, it can be considered from the other side that e is independent part of objective variable (left side term: x, y, x_1, x_2). This part only depends on x, y, x_1, $and\ x_2$. When we express independent part of x as $\check{x}$

When $p = 3$

$$\check{x} = e = x - \alpha_{x/z}z = x - \frac{SS_{zx}}{SS_{zz}}z$$

$$\check{y} = \mathrm{e} = y - \alpha_{y/z}z = y - \frac{SS_{yz}}{SS_{zz}}z$$

When $p \geq 4$

$$\widecheck{x_1} = x_1 - \left(\alpha_{x_1/x_3}x_3 + \cdots + \alpha_{x_1/x_p}x_p\right) = x_1 - \left(\frac{SS_{13}}{SS_{33}}x_3 + \cdots + \frac{SS_{13}}{SS_{pp}}x_p\right)$$

$$\widecheck{x_2} = x_2 - \left(+\alpha_{x_2/x_3}x_3 + \cdots + \alpha_{x_2/x_p}x_p\right) = x_2 - \left(\frac{SS_{23}}{SS_{33}}x_3 + \cdots + \frac{SS_{2p}}{SS_{pp}}x_p\right)$$

Firstly, we consider case of $p = 3$

$$\check{x} = x - \alpha_{x/z}z = x - \frac{SS_{zx}}{SS_{zz}}z$$

$$\check{y} = y - \alpha_{y/z}z = y - \frac{SS_{yz}}{SS_{zz}}z$$

$\alpha_{x/z}$: regression coeffcient of z in simple linear regression between x and z.

$\alpha_{y/z}$: regression coeffcient of z in simple linear regression between y and z.

We denote partial correlation coefficient between x and y as $\mathrm{r}_{xy/rest}$.

The meaning of partial correlation coefficient is correlation coefficient after removing influence of the other variables. In the case when there are three variables, x, y and z,

$$\mathrm{r}_{xy|rest} = \mathrm{r}_{xy|z}$$

This is simple correlation between $\check{x}$ and $\check{y}$

$$\mathrm{r}_{xy|z} = r_{\check{x}\check{y}} = \frac{SS_{\check{x}\check{y}}}{\sqrt{SS_{\check{x}\check{x}}}\sqrt{SS_{\check{y}\check{y}}}}.$$

This is the solution when there three variables. In actual operation, we should calculate

$$SS_{\check{x}\check{y}} = \sum_{k=1}^{n}\left(x_k - \frac{SS_{zx}}{SS_{zz}}z_k\right)\left(y_k - \frac{SS_{yz}}{SS_{zz}}z_k\right)$$

$$SS_{\check{x}\check{x}} = \sum_{k=1}^{n}\left(x_k - \frac{SS_{zx}}{SS_{zz}}z_k\right)^2$$

$$SS_{\check{y}\check{y}} = \sum_{k=1}^{n}\left(y_k - \frac{SS_{zy}}{SS_{zz}}z_k\right)^2$$

Calculation

$$SS_{\check{x}\check{y}} = \sum_{k=1}^{n}\left(x_k - \frac{SS_{zx}}{SS_{zz}}z_k\right)\left(y_k - \frac{SS_{yz}}{SS_{zz}}z_k\right)$$

$$= \sum_{k=1}^{n} x_k y_k - \sum_{k=1}^{n}\frac{SS_{zx}}{SS_{zz}}y_k z_k - \sum_{k=1}^{n}\frac{SS_{yz}}{SS_{zz}}x_k z_k + \sum_{k=1}^{n}\frac{SS_{zx}SS_{yz}}{SS_{zz}^{\ 2}}z_k^{\ 2}$$

$$= SS_{xy} + \frac{SS_{zx}SS_{yz}}{SS_{zz}} - \frac{SS_{zx}SS_{yz}}{SS_{zz}} - \frac{SS_{xz}SS_{yz}}{SS_{zz}}$$

$$= SS_{xy} - \frac{SS_{xz}SS_{yz}}{SS_{zz}}$$

$$SS_{\check{x}\check{x}} = \sum_{k=1}^{n}\left(x_k - \frac{SS_{zx}}{SS_{zz}}z_k\right)^2$$

$$= \sum_{k=1}^{n} x_k^{\ 2} - 2\sum_{k=1}^{n}\frac{SS_{zx}}{SS_{zz}}z_k x_k + \sum_{k=1}^{n}\frac{SS_{zx}^{\ 2}}{SS_{zz}^{\ 2}}z_k^{\ 2}$$

$$= SS_{xx} - 2\frac{SS_{zx}^{\ 2}}{SS_{zz}} + \frac{SS_{zx}^{\ 2}}{SS_{zz}}$$

$$= SS_{xx} - \frac{SS_{zx}^{\ 2}}{SS_{zz}}$$

Similarly,

$$SS_{\check{y}\check{y}} = SS_{yy} - \frac{SS_{yz}^{\ 2}}{SS_{zz}}$$

Then

$$\mathrm{r}_{xy|z} = r_{\check{x}\check{y}} = \frac{SS_{xy} - \frac{SS_{xz}SS_{yz}}{SS_{zz}}}{\sqrt{SS_{xx} - \frac{SS_{zx}^{\ 2}}{SS_{zz}}}\sqrt{SS_{yy} - \frac{SS_{yz}^{\ 2}}{SS_{zz}}}}$$

$$= \frac{SS_{xy}SS_{zz} - SS_{xz}SS_{yz}}{\sqrt{SS_{xx}SS_{zz} - SS_{zx}^{\ 2}}\sqrt{SS_{yy}SS_{zz} - SS_{yz}^{\ 2}}}$$

This is the solution of partial correlation of three variables. However, we can express the solution more simply using our notation system in this text.

Inverse matrix of correlation matrix

$$\boldsymbol{R}^{-1} = \frac{1}{|\boldsymbol{R}|}\begin{pmatrix} \frac{SS_{yy}SS_{zz} - {SS_{yz}}^2}{SS_{yy}SS_{zz}} & \frac{SS_{yz}SS_{zx} - SS_{xy}SS_{zz}}{\sqrt{SS_{yy}}\sqrt{SS_{xx}}SS_{zz}} & \frac{SS_{xy}SS_{yz} - SS_{xz}SS_{yy}}{\sqrt{SS_{xx}}\sqrt{SS_{zz}}SS_{yy}} \\ \frac{SS_{yz}SS_{zx} - SS_{xy}SS_{zz}}{\sqrt{SS_{yy}}\sqrt{SS_{xx}}SS_{zz}} & \frac{SS_{zz}SS_{xx} - {SS_{zx}}^2}{SS_{zz}SS_{xx}} & \frac{SS_{xy}SS_{zx} - SS_{yz}SS_{xx}}{\sqrt{SS_{yy}}\sqrt{SS_{zz}}SS_{xx}} \\ \frac{SS_{xy}SS_{yz} - SS_{xz}SS_{yy}}{\sqrt{SS_{xx}}\sqrt{SS_{zz}}SS_{yy}} & \frac{SS_{xy}SS_{zx} - SS_{yz}SS_{xx}}{\sqrt{SS_{yy}}\sqrt{SS_{zz}}SS_{xx}} & \frac{SS_{xx}SS_{yy} - {SS_{xy}}^2}{SS_{xx}SS_{yy}} \end{pmatrix}$$

$$= \begin{pmatrix} r^{xx} & r^{xy} & r^{xz} \\ r^{yx} & r^{yy} & r^{yz} \\ r^{zx} & r^{yz} & r^{zz} \end{pmatrix}$$

$$\mathrm{r}_{xy/z} = \frac{SS_{xy}SS_{zz} - SS_{xz}SS_{yz}}{\sqrt{SS_{xx}SS_{zz} - {SS_{zx}}^2}\sqrt{SS_{yy}SS_{zz} - {SS_{yz}}^2}}$$

$$= \frac{-(SS_{yz}SS_{zx} - SS_{xy}SS_{zz})}{\sqrt{SS_{xx}SS_{zz} - {SS_{zx}}^2}\sqrt{SS_{yy}SS_{zz} - {SS_{yz}}^2}}$$

$$= \frac{-(SS_{yz}SS_{zx} - SS_{xy}SS_{zz})}{\sqrt{SS_{yy}}\sqrt{SS_{xx}}SS_{zz}} \cdot \frac{\sqrt{SS_{zz}SS_{xx}}}{\sqrt{SS_{xx}SS_{zz} - {SS_{zx}}^2}} \cdot \frac{\sqrt{SS_{yy}SS_{zz}}}{\sqrt{SS_{yy}SS_{zz} - {SS_{yz}}^2}}$$

$$= \frac{-(SS_{yz}SS_{zx} - SS_{xy}SS_{zz})}{\sqrt{SS_{yy}}\sqrt{SS_{xx}}SS_{zz}} \cdot \sqrt{\frac{SS_{zz}SS_{xx}}{SS_{xx}SS_{zz} - {SS_{zx}}^2}} \cdot \sqrt{\frac{SS_{yy}SS_{zz}}{SS_{yy}SS_{zz} - {SS_{yz}}^2}}$$

$$= -r^{xy} \cdot \sqrt{\frac{1}{r^{yy}}}\sqrt{\frac{1}{r^{xx}}}$$

$$= \frac{-r^{xy}}{\sqrt{r^{xx}}\sqrt{r^{yy}}}$$

Formula 74

We consider the solution in the case when $p \geq 4$

Logically, the solution is as follow.

$$\mathrm{r}_{ij/rest} = r_{\breve{x}_i\breve{x}_j} = \frac{SS_{\breve{x}_i\breve{x}_j}}{\sqrt{SS_{\breve{x}_i\breve{x}_i}}\sqrt{SS_{\breve{x}_j\breve{x}_j}}}$$

$$\breve{x}_1 = x_1 - \left(\frac{SS_{13}}{SS_{33}}x_3 + \cdots + \frac{SS_{1p}}{SS_{pp}}x_p\right)$$

$$\breve{x}_2 = x_2 - \left(\frac{SS_{23}}{SS_{33}}x_3 + \cdots + \frac{SS_{2p}}{SS_{pp}}x_p\right)$$

$$SS_{\widetilde{x_1}\widetilde{x_2}} = \sum_{k=1}^{n}\left(x_{1k} - \left(\sum_{l=3}^{p}\frac{SS_{1l}}{SS_{ll}}x_{lk}\right)\right)\left(x_{2k} - \left(\sum_{l=3}^{p}\frac{SS_{2l}}{SS_{ll}}x_{lk}\right)\right)$$

$$SS_{\widetilde{x_1}\widetilde{x_1}} = \sum_{k=1}^{n}\left(x_{1k} - \left(\sum_{l=3}^{p}\frac{SS_{1l}}{SS_{ll}}x_{lk}\right)\right)^2$$

$$SS_{\widetilde{x_2}\widetilde{x_2}} = \sum_{k=1}^{n}\left(x_{2k} - \left(\sum_{l=3}^{p}\frac{SS_{2l}}{SS_{ll}}x_{lk}\right)\right)^2$$

$$\mathrm{r}_{x_1x_2|rest} = r_{\widetilde{x_1}\widetilde{x_2}} = \frac{SS_{\widetilde{x_i}\widetilde{x_j}}}{\sqrt{SS_{\widetilde{x_i}\widetilde{x_i}}}\sqrt{SS_{\cdot\widetilde{x_j}\widetilde{x_j}}}}$$

We can arrange the variance and covariance matrix to set target variables to come to first and second row of the matrix without change of sign, because we have to change column with change of row.

So, we can express the solution as follow.

$$\mathrm{r}_{x_ix_j|res} == \frac{SS_{\widetilde{x_i}\widetilde{x_j}}}{\sqrt{SS_{\widetilde{x_i}\widetilde{x_i}}}\sqrt{SS_{\cdot\widetilde{x_j}\widetilde{x_j}}}}$$

This s a form of solution. However, calculation of partial correlation using this formula, it is too complicated. We need more simple expression of the solution for in practical sense.

One possible approach is step by step approach,

When there are for variables x, y, z, w, and we need partial correlation between x and y, we remove z and w step by step.

At first, we calculate correlation matrix.

$$\boldsymbol{R}_{xyzw} = \begin{pmatrix} 1 & r_{xy} & r_{xz} & r_{xw} \\ r_{yx} & 1 & r_{yz} & r_{yw} \\ r_{zx} & r_{zy} & 1 & r_{zw} \\ r_{wx} & r_{wy} & r_{wz} & 1 \end{pmatrix}$$

$$\boldsymbol{R}_{xyzw}{}^{-1} = \frac{1}{|\boldsymbol{R}|}\begin{pmatrix} \begin{vmatrix} 1 & r_{yz} & r_{yw} \\ r_{zy} & 1 & r_{zw} \\ r_{wy} & r_{wz} & 1 \end{vmatrix} & -\begin{vmatrix} r_{yx} & r_{yz} & r_{yw} \\ r_{zx} & 1 & r_{zw} \\ r_{wx} & r_{wz} & 1 \end{vmatrix} & \begin{vmatrix} r_{yx} & 1 & r_{yw} \\ r_{zx} & r_{zy} & r_{zw} \\ r_{wx} & r_{wy} & 1 \end{vmatrix} & -\begin{vmatrix} r_{yx} & 1 & r_{yz} \\ r_{zx} & r_{zy} & 1 \\ r_{wx} & r_{wy} & r_{wz} \end{vmatrix} \\ -\begin{vmatrix} r_{xy} & r_{xz} & r_{xw} \\ r_{zy} & 1 & r_{zw} \\ r_{wy} & r_{wz} & 1 \end{vmatrix} & \begin{vmatrix} 1 & r_{xz} & r_{xw} \\ r_{zx} & 1 & r_{zw} \\ r_{wx} & r_{wz} & 1 \end{vmatrix} & -\begin{vmatrix} 1 & r_{xy} & r_{xw} \\ r_{zx} & r_{zy} & r_{zw} \\ r_{wx} & r_{wy} & 1 \end{vmatrix} & \begin{vmatrix} 1 & r_{xy} & r_{xz} \\ r_{zx} & r_{zy} & 1 \\ r_{wx} & r_{wy} & r_{wz} \end{vmatrix} \\ \begin{vmatrix} r_{xy} & r_{xz} & r_{xw} \\ 1 & r_{yz} & r_{yw} \\ r_{wy} & r_{wz} & 1 \end{vmatrix} & -\begin{vmatrix} 1 & r_{xz} & r_{xw} \\ r_{yx} & r_{yz} & r_{yw} \\ r_{wx} & r_{wz} & 1 \end{vmatrix} & \begin{vmatrix} 1 & r_{xy} & r_{xw} \\ r_{yx} & 1 & r_{yw} \\ r_{wx} & r_{wy} & 1 \end{vmatrix} & -\begin{vmatrix} 1 & r_{xy} & r_{xz} \\ r_{yx} & 1 & r_{yz} \\ r_{wx} & r_{wy} & r_{wz} \end{vmatrix} \\ -\begin{vmatrix} r_{xy} & r_{xz} & r_{xw} \\ 1 & r_{yz} & r_{yw} \\ r_{zy} & 1 & r_{zw} \end{vmatrix} & \begin{vmatrix} 1 & r_{xz} & r_{xw} \\ r_{yx} & r_{yz} & r_{yw} \\ r_{zx} & 1 & r_{zw} \end{vmatrix} & -\begin{vmatrix} 1 & r_{xy} & r_{xw} \\ r_{yx} & 1 & r_{yw} \\ r_{zx} & r_{wy} & r_{zw} \end{vmatrix} & \begin{vmatrix} 1 & r_{xy} & r_{xz} \\ r_{yx} & 1 & r_{yz} \\ r_{zx} & r_{wy} & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} r^{xx} & r^{xy} & r^{xz} & r^{xw} \\ r^{yx} & r^{yy} & r^{yz} & r^{yw} \\ r^{zx} & r^{zy} & r^{zz} & r^{zw} \\ r^{wx} & r^{wy} & r^{wz} & r^{ww} \end{pmatrix}$$

$$r^{xx} = \frac{1}{|\boldsymbol{R}|}\begin{vmatrix} 1 & r_{yz} & r_{yw} \\ r_{zy} & 1 & r_{zw} \\ r_{wy} & r_{wz} & 1 \end{vmatrix} = \frac{1}{|\boldsymbol{R}|}\left(1 + 2r_{yz}r_{zw}r_{wy} - r_{yz}{}^{2} - r_{yw}{}^{2} - r_{zw}\right)$$

$$r^{yy} = \frac{1}{|\boldsymbol{R}|}\begin{vmatrix} 1 & r_{xz} & r_{xw} \\ r_{zx} & 1 & r_{zw} \\ r_{wx} & r_{wz} & 1 \end{vmatrix} = \frac{1}{|\boldsymbol{R}|}\left(1 + 2r_{xz}r_{zw}r_{wx} - r_{xz}{}^{2} - r_{xw}{}^{2} + r_{zw}{}^{2}\right)$$

$$r^{xy} = -\frac{1}{|\boldsymbol{R}|}\begin{vmatrix} r_{yx} & r_{yz} & r_{yw} \\ r_{zx} & 1 & r_{zw} \\ r_{wx} & r_{wz} & 1 \end{vmatrix} = -\frac{1}{|\boldsymbol{R}|}\left(r_{yx} + r_{yz}r_{zw}r_{wx} + r_{yw}r_{zx}r_{wz} - \left(r_{yx}r_{zw}{}^{2} + r_{yz}r_{zx} + r_{yw}r_{wx}\right)\right)$$

$$\therefore \frac{-r^{xy}}{\sqrt{r^{xx}}\sqrt{r^{yy}}} = \frac{r_{yx} + r_{yz}r_{zw}r_{wx} + r_{yw}r_{zx}r_{wz} - \left(r_{yx}r_{zw}{}^{2} + r_{yz}r_{zx} + r_{yw}r_{wx}\right)}{\sqrt{\left(1 + 2r_{yz}r_{zw}r_{wy} - r_{yz}{}^{2} - r_{yw}{}^{2} - r_{zw}\right)}\left(1 + 2r_{xz}r_{zw}r_{wx} - r_{xz}{}^{2} - r_{xw}{}^{2} + r_{zw}{}^{2}\right)} \qquad (i)$$

We remove w.

For $r_{xy|w}$

$$\boldsymbol{R}_{xyw} = \begin{pmatrix} 1 & r_{xy} & r_{xw} \\ r_{yx} & 1 & r_{yw} \\ r_{wx} & r_{wy} & 1 \end{pmatrix}$$

(Pay attention, $|\boldsymbol{R}_{xyw}| = r^{zz}$)

$$\boldsymbol{R}_{xyw}{}^{-1} = \frac{1}{|\boldsymbol{R}_{xyw}|}\begin{pmatrix} \begin{vmatrix} 1 & r_{yw} \\ r_{wy} & 1 \end{vmatrix} & -\begin{vmatrix} r_{yx} & r_{yw} \\ r_{wx} & 1 \end{vmatrix} & \begin{vmatrix} r_{yx} & 1 \\ r_{wx} & r_{wy} \end{vmatrix} \\ -\begin{vmatrix} r_{xy} & r_{xw} \\ r_{wy} & 1 \end{vmatrix} & \begin{vmatrix} 1 & r_{xw} \\ r_{wx} & 1 \end{vmatrix} & -\begin{vmatrix} 1 & r_{xy} \\ r_{wx} & r_{wy} \end{vmatrix} \\ \begin{vmatrix} r_{xy} & r_{xw} \\ 1 & r_{yw} \end{vmatrix} & -\begin{vmatrix} 1 & r_{xw} \\ r_{yx} & r_{yw} \end{vmatrix} & \begin{vmatrix} 1 & r_{xy} \\ r_{yx} & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} r_{xy|w}{}^{xx} & r_{xy|w}{}^{xy} & r_{xy|w}{}^{xw} \\ r_{xy|w}{}^{yx} & r_{xy|w}{}^{yy} & r_{xy|w}{}^{yw} \\ r_{xy|w}{}^{wx} & r_{xy|w}{}^{wy} & r_{xy|w}{}^{ww} \end{pmatrix}$$

$$r_{xy|w} = \frac{-r_{xy|w}{}^{xy}}{\sqrt{r_{xy|w}{}^{xx}}\sqrt{r_{xy|w}{}^{yy}}} = \frac{\begin{vmatrix} r_{yx} & r_{yw} \\ r_{wx} & 1 \end{vmatrix}}{\sqrt{\begin{vmatrix} 1 & r_{yw} \\ r_{wy} & 1 \end{vmatrix}}\sqrt{\begin{vmatrix} 1 & r_{xw} \\ r_{wx} & 1 \end{vmatrix}}}$$

$$= \frac{r_{yx} - r_{yw}r_{wx}}{\sqrt{\left(1 - r_{yw}{}^2\right)\left(1 - r_{wx}{}^2\right)}}$$

$$r_{xy|w} = \frac{r_{yx} - r_{yw}r_{wx}}{\sqrt{\left(1 - r_{yw}{}^2\right)\left(1 - r_{wx}{}^2\right)}} \qquad (ii)$$

For $r_{xz/w}$

$$\boldsymbol{R}_{\boldsymbol{xzw}} = \begin{pmatrix} 1 & r_{xz} & r_{xw} \\ r_{zx} & 1 & r_{zw} \\ r_{wx} & r_{wz} & 1 \end{pmatrix}$$

(Pay attention, $|\boldsymbol{R}_{\boldsymbol{xzw}}| = r^{yy}$)

$$\boldsymbol{R}_{\boldsymbol{xzw}}{}^{-1} = \frac{1}{|\boldsymbol{R}_{\boldsymbol{xzw}}|} \begin{pmatrix} \begin{vmatrix} 1 & r_{zw} \\ r_{zw} & 1 \end{vmatrix} & -\begin{vmatrix} r_{zx} & r_{zw} \\ r_{wx} & 1 \end{vmatrix} & \begin{vmatrix} r_{zx} & 1 \\ r_{wx} & r_{wz} \end{vmatrix} \\ -\begin{vmatrix} r_{xz} & r_{xw} \\ r_{wz} & 1 \end{vmatrix} & \begin{vmatrix} 1 & r_{xw} \\ r_{wx} & 1 \end{vmatrix} & -\begin{vmatrix} 1 & r_{xz} \\ r_{wx} & r_{wz} \end{vmatrix} \\ \begin{vmatrix} r_{xz} & r_{xw} \\ 1 & r_{zw} \end{vmatrix} & -\begin{vmatrix} 1 & r_{xw} \\ r_{zx} & r_{zw} \end{vmatrix} & \begin{vmatrix} 1 & r_{xz} \\ r_{zx} & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} r_{xz|w}{}^{xx} & r_{xz|w}{}^{xz} & r_{xz|w}{}^{xw} \\ r_{xz|w}{}^{zx} & r_{xz|w}{}^{zz} & r_{xz|w}{}^{zw} \\ r_{xz|w}{}^{wx} & r_{xz|w}{}^{wz} & r_{xz|w}{}^{ww} \end{pmatrix}$$

$$r_{xz|w} = \frac{-r_{xz|w}{}^{xz}}{\sqrt{r_{xz|w}{}^{xx}}\sqrt{r_{xz|w}{}^{zz}}} = \frac{\begin{vmatrix} r_{zx} & r_{zw} \\ r_{wx} & 1 \end{vmatrix}}{\sqrt{\begin{vmatrix} 1 & r_{zw} \\ r_{wz} & 1 \end{vmatrix}}\sqrt{\begin{vmatrix} 1 & r_{xw} \\ r_{wx} & 1 \end{vmatrix}}}$$

$$= \frac{r_{zx} - r_{zw}r_{wx}}{\sqrt{(1 - r_{zw}{}^2)(1 - r_{wx}{}^2)}}$$

$$r_{xz|w} == \frac{r_{zx} - r_{zw}r_{wx}}{\sqrt{(1 - r_{zw}{}^2)(1 - r_{wx}{}^2)}} \qquad (iii)$$

For $r_{yz/w}$

$$\boldsymbol{R}_{\boldsymbol{yzw}} = \begin{pmatrix} 1 & r_{yz} & r_{yw} \\ r_{zy} & 1 & r_{zw} \\ r_{wy} & r_{wz} & 1 \end{pmatrix}$$

(Pay attention, $|\boldsymbol{R}_{\boldsymbol{yzw}}| = r^{yy}$)

$$\boldsymbol{R}_{\boldsymbol{yzw}}^{-1} = \frac{1}{|\boldsymbol{R}_{\boldsymbol{yzw}}|} \begin{pmatrix} \begin{vmatrix} 1 & r_{zw} \\ r_{wz} & 1 \end{vmatrix} & -\begin{vmatrix} r_{zy} & r_{zw} \\ r_{wy} & 1 \end{vmatrix} & \begin{vmatrix} r_{zy} & 1 \\ r_{wy} & r_{wz} \end{vmatrix} \\ -\begin{vmatrix} r_{yz} & r_{yw} \\ r_{wz} & 1 \end{vmatrix} & \begin{vmatrix} 1 & r_{yw} \\ r_{wy} & 1 \end{vmatrix} & -\begin{vmatrix} 1 & r_{yz} \\ r_{wy} & r_{wz} \end{vmatrix} \\ \begin{vmatrix} r_{yz} & r_{yw} \\ 1 & r_{zw} \end{vmatrix} & -\begin{vmatrix} 1 & r_{yw} \\ r_{zy} & r_{zw} \end{vmatrix} & \begin{vmatrix} 1 & r_{yz} \\ r_{zy} & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} r_{yz|w}{}^{yy} & r_{yz|w}{}^{yz} & r_{yz|w}{}^{yw} \\ r_{yz|w}{}^{zy} & r_{yz|w}{}^{zz} & r_{yz|w}{}^{zw} \\ r_{yz|w}{}^{wy} & r_{yz|w}{}^{wz} & r_{yz|w}{}^{ww} \end{pmatrix}$$

$$r_{yz|w} = \frac{-r_{yz|w}{}^{yz}}{\sqrt{r_{yz|w}{}^{yy}}\sqrt{r_{yz|w}{}^{zz}}} = \frac{\begin{vmatrix} r_{zy} & r_{zw} \\ r_{wy} & 1 \end{vmatrix}}{\sqrt{\begin{vmatrix} 1 & r_{zw} \\ r_{wz} & 1 \end{vmatrix}}\sqrt{\begin{vmatrix} 1 & r_{yw} \\ r_{wy} & 1 \end{vmatrix}}}$$

$$r_{yz|w} = \frac{r_{zy} - r_{zw}r_{wy}}{\sqrt{(1 - r_{zw}{}^2)(1 - r_{yw}{}^2)}} \qquad (iv)$$

Then we make 3 x 3 correlation matrix among $r_{xy/w}$, $r_{xz/w}$ and $r_{yz/w}$

$$\boldsymbol{R}_{\boldsymbol{\breve{x}\breve{y}\breve{z}}} = \begin{pmatrix} 1 & r_{xy/w} & r_{xz/w} \\ r_{xy/w} & 1 & r_{yz/w} \\ r_{xz/w} & r_{yz/w} & 1 \end{pmatrix}$$

$$\boldsymbol{R}_{\boldsymbol{\breve{x}\breve{y}\breve{z}}}^{-1} = \frac{1}{|\boldsymbol{R}_{\boldsymbol{\breve{x}\breve{y}\breve{z}}}|} \begin{pmatrix} \begin{vmatrix} 1 & r_{yz/w} \\ r_{yz/w} & 1 \end{vmatrix} & -\begin{vmatrix} r_{xy/w} & r_{yz/w} \\ r_{xz/w} & 1 \end{vmatrix} & \begin{vmatrix} r_{xy/w} & 1 \\ r_{xz/w} & r_{yz/w} \end{vmatrix} \\ -\begin{vmatrix} r_{xy/w} & r_{xz/w} \\ r_{yz/w} & 1 \end{vmatrix} & \begin{vmatrix} 1 & r_{xz/w} \\ r_{xz/w} & 1 \end{vmatrix} & -\begin{vmatrix} 1 & r_{xy/w} \\ r_{xz/w} & r_{yz/w} \end{vmatrix} \\ \begin{vmatrix} r_{xy/w} & r_{xz/w} \\ 1 & r_{yz/w} \end{vmatrix} & -\begin{vmatrix} 1 & r_{xz/w} \\ r_{xy/w} & r_{yz/w} \end{vmatrix} & \begin{vmatrix} 1 & r_{xy/w} \\ r_{xy/w} & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} r_{xyz/w}{}^{xx} & r_{xyz/w}{}^{xy} & r_{xyz/w}{}^{xz} \\ r_{xyz/w}{}^{yx} & r_{xyz/w}{}^{yy} & r_{xyz/w}{}^{yz} \\ r_{xyz/w}{}^{zx} & r_{xyz/w}{}^{zy} & r_{xyz/w}{}^{zz} \end{pmatrix}$$

$$r_{xy|zw} = r_{\bar{\bar{x}}\bar{\bar{y}}} = \frac{-r_{xyz|w}{}^{xy}}{\sqrt{r_{xyz|w}{}^{xx}}\sqrt{r_{xyz|w}{}^{yy}}} \qquad (v)$$

For $r_{xyz|w}{}^{xx}$

$$r_{xyz|w}{}^{xx} = \frac{\begin{vmatrix} 1 & r_{yz|w} \\ r_{r_{yz|w}} & 1 \end{vmatrix}}{|\boldsymbol{R}_{\tilde{x}\tilde{y}\tilde{z}}|} = \frac{1 - r_{yz|w}{}^2}{|\boldsymbol{R}_{\tilde{x}\tilde{y}\tilde{z}}|}$$

$$= \frac{1}{|\boldsymbol{R}_{\tilde{x}\tilde{y}\tilde{z}}|}\left(1 - \frac{\left(r_{zy} - r_{zw}r_{wy}\right)^2}{(1 - r_{zw}{}^2)(1 - r_{yw}{}^2)}\right)$$

$$= \frac{1 - r_{zw}{}^2 - r_{yw}{}^2 + r_{zw}{}^2 r_{yw}{}^2 - r_{zy}{}^2 + 2r_{zy}r_{zw}r_{wy} - r_{zw}{}^2 r_{yw}{}^2}{|\boldsymbol{R}_{\tilde{x}\tilde{y}\tilde{z}}|(1 - r_{zw}{}^2)(1 - r_{yw}{}^2)}$$

$$= \frac{1 + 2r_{zy}r_{zw}r_{wy} - r_{zw}{}^2 - r_{yw}{}^2 - r_{zy}{}^2}{|\boldsymbol{R}_{\tilde{x}\tilde{y}\tilde{z}}|(1 - r_{zw}{}^2)(1 - r_{yw}{}^2)}$$

$$r_{xyz|w}{}^{xx} = \frac{1 + 2r_{zy}r_{zw}r_{wy} - r_{zw}{}^2 - r_{yw}{}^2 - r_{zy}{}^2}{|\boldsymbol{R}_{\tilde{x}\tilde{y}\tilde{z}}|(1 - r_{zw}{}^2)(1 - r_{yw}{}^2)} \qquad (vi)$$

For $r_{xyz|w}{}^{yy}$

$$r_{xyz|w}{}^{yy} = \frac{\begin{vmatrix} 1 & r_{xz|w} \\ r_{xz|w} & 1 \end{vmatrix}}{|\boldsymbol{R}_{\tilde{x}\tilde{y}\tilde{z}}|}$$

$$= \frac{\begin{vmatrix} 1 & r_{xz|w} \\ r_{xz|w} & 1 \end{vmatrix}}{|\boldsymbol{R}_{\tilde{x}\tilde{y}\tilde{z}}|} = \frac{1 - r_{xz|w}{}^2}{|\boldsymbol{R}_{\tilde{x}\tilde{y}\tilde{z}}|}$$

$$= \frac{1}{|\boldsymbol{R}_{\tilde{x}\tilde{y}\tilde{z}}|}\left(1 - \frac{(r_{zx} - r_{zw}r_{wx})^2}{(1 - r_{zw}{}^2)(1 - r_{wx}{}^2)}\right)$$

$$= \frac{(1 - r_{zw}{}^2)(1 - r_{wx}{}^2) - (r_{zx} - r_{zw}r_{wx})^2}{|\boldsymbol{R}_{\tilde{x}\tilde{y}\tilde{z}}|(1 - r_{zw}{}^2)(1 - r_{wx}{}^2)}$$

$$= \frac{1 - r_{zw}{}^2 - r_{wx}{}^2 + r_{zw}{}^2 r_{wx}{}^2 - (r_{zx}{}^2 - 2r_{zx}r_{zw}r_{wx} + r_{zw}{}^2 r_{wx}{}^2)^2}{|\boldsymbol{R}_{\tilde{x}\tilde{y}\tilde{z}}|(1 - r_{zw}{}^2)(1 - r_{wx}{}^2)}$$

$$= \frac{1 + 2r_{zx}r_{zw} - r_{zw}{}^2 - r_{wx}{}^2 - r_{zx}{}^2}{|\boldsymbol{R}_{\tilde{x}\tilde{y}\tilde{z}}|(1 - r_{zw}{}^2)(1 - r_{wx}{}^2)}$$

$$r_{xyz|w}{}^{yy} = \frac{1 + 2r_{zx}r_{zw} - r_{zw}{}^2 - r_{wx}{}^2 - r_{zx}{}^2}{|\boldsymbol{R}_{\tilde{x}\tilde{y}\tilde{z}}|(1 - r_{zw}{}^2)(1 - r_{wx}{}^2)} \qquad (vii)$$

For $r_{xyz|w}{}^{xy}$

$$r_{xyz|w}{}^{xy} = \frac{-\begin{vmatrix} r_{xy|w} & r_{yz|w} \\ r_{xz|w} & 1 \end{vmatrix}}{|\boldsymbol{R}_{\tilde{x}\tilde{y}\tilde{z}}|} = \frac{-\left(r_{xy|w} - r_{yz|w}r_{xz|w}\right)}{|\boldsymbol{R}_{\tilde{x}\tilde{y}\tilde{z}}|}$$

$$= \frac{-\left(\frac{r_{yx} - r_{yw}r_{wx}}{\sqrt{\left(1 - r_{yw}{}^2\right)(1 - r_{wx}{}^2)}} - \frac{r_{zy} - r_{zw}r_{wy}}{\sqrt{(1 - r_{zw}{}^2)\left(1 - r_{yw}{}^2\right)}}\frac{r_{zx} - r_{zw}r_{wx}}{\sqrt{(1 - r_{zw}{}^2)(1 - r_{wx}{}^2)}}\right)}{|\boldsymbol{R}_{\tilde{x}\tilde{y}\tilde{z}}|}]$$

$$= \frac{-\left(\left(r_{yx} - r_{yw}r_{wx}\right)\left(1 - r_{zw}{}^2\right) - \left(r_{zy} - r_{zw}r_{wy}\right)\left(r_{zx} - r_{zw}r_{wx}\right)\right)}{\left|\boldsymbol{R}_{\check{x}\check{y}\check{z}}\right|\left(1 - r_{zw}{}^2\right)\sqrt{\left(1 - r_{yw}{}^2\right)\left(1 - r_{wx}{}^2\right)}}$$

$$= \frac{-\left(r_{yx} - r_{yw}r_{wx} - r_{zw}{}^2 + r_{yw}r_{wx}r_{zw}{}^2 - \left(r_{zy}r_{zx} - r_{zw}r_{wy}r_{zx} - r_{zy}r_{zw}r_{wx} + r_{yw}r_{wx}r_{zw}{}^2\right)\right)}{\left|\boldsymbol{R}_{\check{x}\check{y}\check{z}}\right|\left(1 - r_{zw}{}^2\right)\sqrt{\left(1 - r_{yw}{}^2\right)\left(1 - r_{wx}{}^2\right)}}$$

$$\frac{-\left(r_{yx} + r_{zw}r_{wy}r_{zx} + r_{zy}r_{zw}r_{wx} - r_{yw}r_{wx} - r_{zw}{}^2 - r_{zy}r_{zx}\right)}{\left|\boldsymbol{R}_{\check{x}\check{y}\check{z}}\right|\left(1 - r_{zw}{}^2\right)\sqrt{\left(1 - r_{yw}{}^2\right)\left(1 - r_{wx}{}^2\right)}}$$

$$r_{xyz|w}{}^{xy} = \frac{-\left(r_{yx} + r_{zw}r_{wy}r_{zx} + r_{zy}r_{zw}r_{wx} - r_{yw}r_{wx} - r_{zw}{}^2 - r_{zy}r_{zx}\right)}{\left|\boldsymbol{R}_{\check{x}\check{y}\check{z}}\right|\left(1 - r_{zw}{}^2\right)\sqrt{\left(1 - r_{yw}{}^2\right)\left(1 - r_{wx}{}^2\right)}} \qquad (viii)$$

Summary

(i)
$$\frac{-r^{xy}}{\sqrt{r^{xx}}\sqrt{r^{yy}}} = \frac{r_{yx} + r_{yz}r_{zw}r_{wx} + r_{yw}r_{zx}r_{wz} - \left(r_{yx}r_{zw}{}^2 + r_{yz}r_{zx} + r_{yw}r_{wx}\right)}{\sqrt{\left(1 + 2r_{yz}r_{zw}r_{wy} - r_{yz}{}^2 - r_{yw}{}^2 - r_{zw}\right)\left(1 + 2r_{xz}r_{zw}r_{wx} - r_{xz}{}^2 - r_{xw}{}^2 + r_{zw}{}^2\right)}}$$

(v)
$$r_{xy/zw} = r_{\bar{\bar{x}}\bar{\bar{y}}} = \frac{-r_{xyz/w}{}^{xy}}{\sqrt{r_{xyz/w}{}^{xx}}\sqrt{r_{xyz/w}{}^{yy}}}$$

(vi)
$$r_{xyz|w}{}^{xx} = \frac{1 + 2r_{zy}r_{zw}r_{wy} - r_{zw}{}^2 - r_{yw}{}^2 - r_{zy}{}^2}{\left|\boldsymbol{R}_{\check{x}\check{y}\check{z}}\right|\left(1 - r_{zw}{}^2\right)\left(1 - r_{yw}{}^2\right)}$$

(vii)
$$r_{xyz|w}{}^{yy} = \frac{1 + 2r_{zx}r_{zw} - r_{zw}{}^2 - r_{wx}{}^2 - r_{zx}{}^2}{\left|\boldsymbol{R}_{\check{x}\check{y}\check{z}}\right|\left(1 - r_{zw}{}^2\right)\left(1 - r_{wx}{}^2\right)}$$

$(viii)$
$$r_{xyz|w}{}^{xy} = \frac{-\left(r_{yx} + r_{zw}r_{wy}r_{zx} + r_{zy}r_{zw}r_{wx} - r_{yw}r_{wx} - r_{zw}{}^2 - r_{zy}r_{zx}\right)}{\left|\boldsymbol{R}_{\check{x}\check{y}\check{z}}\right|\left(1 - r_{zw}{}^2\right)\sqrt{\left(1 - r_{yw}{}^2\right)\left(1 - r_{wx}{}^2\right)}}$$

Put $(vi), (vii)$ and $(viii)$ in (v)

$$r_{xy|zw} = \frac{-r_{xyz|w}{}^{xy}}{\sqrt{r_{xyz|w}{}^{xx}}\sqrt{r_{xyz|w}{}^{yy}}} = \frac{r_{yx} + r_{zw}r_{wy}r_{zx} + r_{zy}r_{zw}r_{wx} - r_{yw}r_{wx} - r_{zw}{}^2 - r_{zy}r_{zx}}{\sqrt{1 + 2r_{zy}r_{zw}r_{wy} - r_{zw}{}^2 - r_{yw}{}^2 - r_{zy}{}^2}\sqrt{1 + 2r_{zx}r_{zw} - r_{zw}{}^2 - r_{wx}{}^2 - r_{zx}{}^2}} \qquad (ix)$$

Put (i) in (ix)

$$r_{xy|zw} = \frac{-r^{xy}}{\sqrt{r^{xx}}\sqrt{r^{yy}}}$$

Conclusion

$$r_{x_1x_2/x_3x_4} = r_{x_1x_2/rest} = \frac{-r^{x_1x_2}}{\sqrt{r^{x_1x_1}}\sqrt{r^{x_2x_2}}}$$

In the case when number of variables is p, we can express partial correlation of all other combination of remained variables after removing pth variable by combination of cofactor, and we can make $\times (p-1)$ matrix. Then we can remove $(p-1)^{\text{th}}$ variable by same method. Repeating this we can proof

$$r_{x_1x_2/x_3\cdots x_p} = r_{x_1x_2/rest} = \frac{-r^{x_1x_2}}{\sqrt{r^{x_1x_1}}\sqrt{r^{x_2x_2}}}$$

For this purpose, wee need to shift the row and column of target variables to first and second row and column as in figure 66.

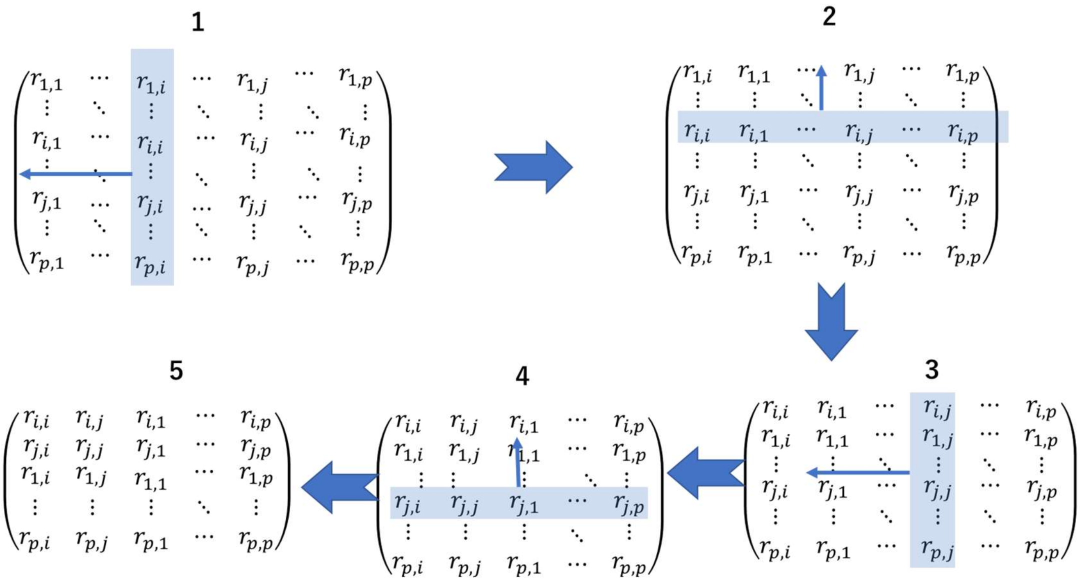

Figure 66. Shifting of row and column of variables for partial correlation analysis of target variables.

Generally, we need change the sing of matrix with shift of row and column. However, we do not need consideration of sign, because

$$sing\ (R) = (-1)^{2(i-1)}(-1)^{2(j-1)}$$

$$\begin{pmatrix} r_{1,1} & \cdots & r_{1,i} & \cdots & r_{1,j} & \cdots & r_{1,p} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ r_{i,1} & \cdots & r_{i,i} & \cdots & r_{i,j} & \cdots & r_{i,p} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ r_{j,1} & \cdots & r_{j,i} & \cdots & r_{j,j} & \cdots & r_{j,p} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ r_{p,1} & \cdots & r_{p,i} & \cdots & r_{p,j} & \cdots & r_{p,p} \end{pmatrix}$$

$$
= \begin{pmatrix}
r_{i,i} & r_{i,j} & r_{i,1} & \cdots & r_{i,i-1} & r_{i,i+1} & \cdots & r_{i,j-1} & r_{i,j+1} & \cdots & r_{i,p} \\
r_{j,i} & r_{j,j} & r_{j,1} & \cdots & r_{j,i-1} & r_{j,i+1} & \cdots & r_{j,j-1} & r_{j,j+1} & \cdots & r_{j,p} \\
r_{1,i} & r_{1,j} & r_{1,1} & \cdots & r_{1,i-1} & r_{1,i+1} & \cdots & r_{1,j-1} & r_{1,j+1} & \cdots & r_{1,p} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
r_{i-1,i} & r_{i-1,j} & r_{i-1,1} & \cdots & r_{i-1,i-1} & r_{i-1,i+1} & \cdots & r_{i-1,j-1} & r_{i-1,j+1} & \cdots & r_{i-1,p} \\
r_{i+1,i} & r_{i+1,j} & r_{i+1,1} & \cdots & r_{i+1,i-1} & r_{i+1,i+1} & \cdots & r_{i+1,j-1} & r_{i+1,j+1} & \cdots & r_{i+1,p} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
r_{j-1,i} & r_{j-1,j} & r_{j-1,1} & \cdots & r_{j-1,i-1} & r_{j-1,i+1} & \cdots & r_{j-1,j-1} & r_{j-1,j+1} & \cdots & r_{j-1,p} \\
r_{j+1,i} & r_{j+1,j} & r_{j+1,1} & \cdots & r_{j+1,i-1} & r_{j+1,i+1} & \cdots & r_{j+1,j-1} & r_{j+1,j+1} & \cdots & r_{j+1,p} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
r_{p,i} & r_{p,j} & r_{p,1} & \cdots & r_{p,i-1} & r_{p,i+1} & \cdots & r_{p,j-1} & r_{p,j+1} & \cdots & r_{p,p}
\end{pmatrix}
$$

Actually, we don't need to shift the row and line, because we can identify the cofactor of each variables in original inverse matrix.

$$
\begin{pmatrix}
r^{1,1} & \cdots & r^{1,i} & \cdots & r^{1,j} & \cdots & r^{1,p} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
r^{i,1} & \cdots & r^{i,i} & \cdots & r^{i,j} & \cdots & r^{i,p} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
r^{j,1} & \cdots & r^{j,i} & \cdots & r^{j,j} & \cdots & r^{j,p} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
r^{p,1} & \cdots & r^{p,i} & \cdots & r^{p,j} & \cdots & r^{p,p}
\end{pmatrix}
$$

Conclusion

$$
\mathrm{r}_{ij/rest} == \frac{-r^{ij}}{\sqrt{r^{ii}}\sqrt{r^{jj}}}
$$